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Coincidence and common fixed points results by using (CLRF)-property

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Abstract
The aim of this paper is to obtain some coincidence and common fixed point theorems by using (CLRF)-property for hybrid pairs of mappings in b-metric spaces. These results improve, extend and generalize the corresponding results in the literature.

Keywords: Coincidence point, common fixed point, (CLRF)-property, b-metric spaces

Introduction
In 1993, Czerwik [11] introduced the notion of b-metric spaces which generalized the concept of metric spaces. Many authors worked on fixed points of multivalued mappings in different directions in these spaces (for more details see) [2, 3, 7, 9]. In 1996, Jungck [4] defined the notion of weakly compatible mappings in metric spaces and proved some common fixed point theorems for such mappings. We can consider [5, 6] to illustrate the relation that compatible mappings are weakly compatible, but converse is not true.
Aamri and Moutawakil [8] in 2002, defined the idea of (E. A) Property for self-mappings which contained the class of non-compatible mappings in metric spaces. The (E. A) property requires the completness (closedness) for the existence of the fixed point in the underlying subspace. To relaxes the requirement of completness (closedness), very first common limit range property with respect to mapping f ((CLRf)-property) is introduced by Sintunaravat and Kumam [15] regarding fuzzy metric space after that this property is used in many other spaces which showed the superiority of (CLRf)-property than (E. A) Property. In this paper we prove common and coincidence fixed point theorems for hybrid mappings along with (CLRf)- property in the setting of complete b-metric spaces.

Mathematical Preliminaries
The following are the concepts from set valued analysis which we shall use in this paper. Let (X, d) be a metric space. Then
B(X) = {A: A is a non-empty bounded subset of X},
CL(X) = {A: A is a non-empty closed subset of X} and
CB(X) = {A: A is a non-empty closed and:bounded subset of X}.

Let T be a multivalued mapping of X in to CB(X) and f be a self Mapping of X. Then the pair (f, S) is said to be a hybrid pair. An element x:∈ X is said to be a coincidence point of T: X → CB(X) and f: X → X if fx ∈ Tx. We denote C (f, T) = {x ∈ X: fx ∈ Tx}, the set
of coincidence point of T and f.

Definition 1 [15]: Let f, g: X → X be two self-mappings. Then the pair (f, g) is said to satisfy common limit range property with respect to the mapping f (CLRf) [15] if there exists a sequence {xn} in X such that:lim fxn =:lim gxn = fu, for some:u ∈ X.

Definition 2 [10]: Let f: X → X and T: X → CB(X) be a single valued and multivalued mapping respectively. Then a hybrid pair of mappings (f, T) is said to satisfy common limit range property with respect to the mapping f (CLRf) [10] if there exists a sequence {xn} in X
such that lim fxn = fu ∈ A =:lim Txn, for some u ∈ X and A∈ CB(X).

Definition 3 [11]: Let X be a non empty set and let s ≥ 1 be a given real number. A function d: X × X → R+ is said to be a b-metric if and only if for all x, y, z ∈ X the following conditions are satisfied:
d (x, y) = 0 if and only if:x = y;
d (x, y) = d(y, x):for all x, y ∈ X;
d (x, y) ≤ s[d(x, z) + d(z, y)] for all x, y, z ∈ X.

Then (X, d, s) is called a b-metric space.
Note that a (usual) metric space is evidently b-metric space. However, Czerwik [11, 12, 13] has shown that a b-metric on X need not be a metric on X. In following example, Singh and Prasad [14] proved that a b-metric on X need not be a metric on X.

Example 4 [14]: Consider the set X = [0, 1] endowed with the function d: X × X → R+ defined by d(x, y) = |x – y|2 for all x, y ∈ X. Clearly, (X, d) is a b-metric space with s = 2, but it is not a metric space.

Example 5 [14]: Let X = {a, b, c} and d (a, c) = d(c, a) = m ≥ 2, d (a, b) = d(b, c) = d(b, a) = d(c, b) = 1 and d(a, a) = d(b, b) = d(c, c) = 0. Then,

$$d(x, y) \quad \frac{m}{2}[d(x, z) + d(z, y)]$$

For all x, y, z ∈ X. If m > 2, the triangle inequality does not hold.

Definition 6 [11]: Let (X, d, s) be a b-metric space. Then a sequence {xn} in X is called:

1. Convergent if and only if there exist x ∈ X such that d

(xn, x) →0 as n → ∞.
2. Cauchy if and only if d (xn, xm) →0 as n, m → ∞ .
3. Complete if and only if every Cauchy sequence is convergent.

Definition 7: Let (X, d, s) be a b-metric space. For A, B ∈ CB(X) and x ∈ X, define the function H: CB(X) × CB(X) →R+ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

Where δ(A, B) = sup{d(a, B): a ∈ A}, δ(B, A) = sup{d(b, A): b ∈ B},

$$\text{with } d(x, A) = \inf\{d(x, a), a \in A\}.$$

Note that H is called the Hausdorff b-metric induced by the b-metric d.
To prove our results we need the following class of functions.

Remarks: Let s ≥ 1 be a real number, we denote Ψs the family of continuous monotone increasing functions in b-metric space, φ: [0, ∞) → [0, ∞) such that

$$\sum_{n=0}^{\infty} s^n \varphi^n(t) < +\infty \text{ for each } t > 0,$$

Where φn denotes n-th iterate of the function φ. It is well known that φ (t) < t for all t > 0 and φ (0) = 0 for t = 0. An example of function φ ∈ Ψs is given by $\varphi(t) = \frac{ct}{s}$ for all t ≥ 0, where c∈ (0, 1).

Common Fixed Points for Mappings with the (CLR)-Property

The following is the definition of (CLRf)-property for two hybrid pairs of single-valued and multivalued mappings in metric spaces.

Definition 1 [1]: Let (X, d) be a metric space. Two single-valued mappings f, g: X → X and two multivalued mappings P, Q: X → CB(X) are said to satisfy the common limit in the range of f (shortly, the (CLRf)-property) if there exists two sequences {xn} and {yn} in X and A, B ∈ CB(X) such that

$$\lim_{n\to+\infty} Px_n = A, \quad \lim_{n\to+\infty} Qy_n = B, \text{ and}$$
$$\lim_{n\to+\infty} gx_n = \lim_{n\to+\infty} gy_n = fu \in A \cap B$$

for some u ∈ X.

Example 2 [1]: Let X = [1, ∞) with the usual metric. Define two single-valued mappings f, g: X → X and two multivalued mappings P, Q: X → CB(X) by

$$fx = 2 + \frac{x}{3},\ gx = 2 + \frac{x}{2},\quad Sx = [1, x + 2] \text{ and } Tx = [3, 3 + \frac{x}{2}],$$

For all x ∈ X, respectively. Then the mappings f and Q satisfy the (CLRf)-property for the sequence {xn} and {yn} defined by $x_n = 3 + \frac{1}{n}$ and $y_n = 2 + \frac{1}{n}$ for each n ≥ 1, respectively.

Indeed, we have

$$\lim_{n\to+\infty} Px_n = [1, 5] = A,\ \lim_{n\to+\infty} Qy_n = [3, 4] = B, \text{ and}$$
$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gy_n = 3 = f(3) = A \cap B$$

Therefore, the pairs (P, f) an f(Q, g) satisfy the (CLRf)-property. Now, we prove the main results in this section.

Theorem 3: Let (X, d, s) be a b-metric space. Let f, g: X → X be two single-valued mappings and P, Q: X → B(X) be two multi-valued mappings satisfying the following conditions:

A. the pair (P, f) and (Q, g) satisfy the (CLRf)-property,
B. for all x, y ∈ X,

$$H(Px, Ty) \le \varphi\Big(\max\Big\{d(fx, gy), d(fx, Px), d(gy, Qy), \frac{1}{2s}[d(fx, Qy) + d(gy, Qx)], \frac{d(fx, Px)(1+d(gy, Qy))}{1+d(fx, gy)}, \frac{d(fx, Qy)(1+d(gy, Px))}{1+d(fx, gy)}\Big\}\Big)$$

If f(X) and g(X) are subsets of X, then we have the following:

1. f and P have a coincidence point,
2. g and Q have a coincidence point,
3. f and P have a common fixed point provided that f and P are weakly compatible at v and ffv = fv for any v ∈ C(f, P),
4. g and Q have a common fixed point provided that g and Q are weakly compatible at v and ggv = gv for any v ∈ C(g, Q),
5. f, g, P, Q have a common point provided that both (3) and (4) are true.

Proof: Since the pairs (P, f) and (Q, g) satisfy the (CLRf)-property, then there exist two sequences {xn}and {yn} in X such that

$$\lim_{n\to+\infty} Px_n = A,\ \lim_{n\to+\infty} Qy_n = B,\ \lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gy_n = fu \in A\cap B$$

For some u ∈X. Now, we show that gw ∈ Qw. In fact, suppose that Gw ∉Qw. Then, using the condition (b) with x = xn and y = w, we have

$$H(Px_n, Qw) \le \varphi\Big(\max\Big\{d(fx_n, gw), d(fx_n, Px_n), d(gw, Qw), \frac{1}{2s}[d(fx_n, Qw) + d(gw, Px_n)], \frac{d(fx_n, Px_n)(1+d(gw, Qw))}{1+d(fx_n, gw)}, \frac{d(fx_n, Qw)(1+d(gw, Px_n))}{1+d(fx_n, gw)}\Big\}\Big)$$

for all n ∈ ℕ. Taking the limit n → ∞, we obtain

$$H(A, Qw) \le \varphi\Big(\max\Big\{d(fv, gw), d(fv, A), d(gw, Qw), \frac{1}{2s}[d(fv, Qw) + d(gw, A)], \frac{d(fv, A)(1+d(gw, Qw))}{1+d(fv, gw)}, \frac{d(fv, Qw)(1+d(gw, A))}{1+d(fv, gw)}\Big\}\Big)$$
$$= \varphi(\max\{0, 0, 0, 0, 0\} = 0.$$

Since gw ∈ A, it follows from the definition of

Hausdorff metric that

$d(gw, Qw) \le H(A, Qw) = 0,$

which is a contradiction and so gw $\in$ Qw. On the other hand, by the condition (b) again, we have

$$H(Pv, Qy_n) \le \varphi\left(\max\left\{d(fv, gy_n), d(fv, Pv), d(gy_n, Qy_n), \frac{1}{2s}[d(fv, Qy_n) + d(gy_n, Pv)], \frac{d(fv,Pv)(1+d(gy_n,Qy_n))}{1+d(fv,gy_n)}, \frac{d(fv,Qy_n)(1+d(gy_n,Pv))}{1+d(fv,gy_n)}\right\}\right)$$

for all n $\in \mathbb{N}$. Similarly, by taking the limit $n \to \infty$, we obtain

$$H(Pv, B) \le \varphi\left(\max\left\{d(fv, gw), d(fv, Pv), d(gw, B), \frac{1}{2s}[d(fv, B) + d(gw, Pv)], \frac{d(fv,Pv)(1+d(gw,B))}{1+d(fv,fu)}, \frac{d(fv,B)(1+d(gw,Pv))}{1+d(fv,fu)}\right\}\right)$$

$$= \varphi(\max\{0, 0, 0, 0, 0\}) = 0.$$

Since fv $\in$ B, it follows from the definition of Hausdorff metric that

$d(fv, Pv) \le H(B, Pv) = 0,$

which is a contradiction and so fv $\in$ Pv. Thus the mappings f, P have a coincidence point v and g, Q have a coincidence point w. Futhermore, by virtue of the condition (b), we obtain ffv = fv and ffv $\in$ Pfv. Thus u = fu $\in$ Pu. This proves (3). A similar argument proves (4). Thus (5) holds immediately. This completes the proof.

Corollary 4: Let (X, d, s) be a b-metric space. Let f: X $\to$ X be a single-valued mapping and P: X $\to$ B(X) be a multi-valued mapping satisfying the following conditions:

a. the pair (P, f) satisfy the (CLRf)-property,
b. for all x, y $\in$ X,

$$H(Px, Py) \le \varphi\left(\max\left\{d(fx, fy), d(fx, Px), d(fy, Py), \frac{1}{2s}[d(fx, Py) + d(fy, Px)], \frac{d(fx,Px)(1+d(fy,Py))}{1+d(fx,fy)}, \frac{d(fx,Py)(1+d(fy,Px))}{1+d(fx,fy)}\right\}\right).$$

If f(X) is subsets of X, then we have the following:

1. f and P have a coincidence point,
2. f and P have a common fixed point provided that f and P are weakly compatible at v and ffv = fv for any v $\in$ C(f, P)

Corollary 5: Let (X, d, s) be a b-metric space. Let f, g, P, Q: X$\to$X be four single-valued mappings satisfying the following conditions:

a. the pair (P, f) and (Q, g) satisfy the (CLRf)-property,
b. for all x, y $\in$ X,

$$d(Px, Qy) \le \varphi\left(\max\left\{d(fx, gy), d(fx, Px), d(gy, Qy), \frac{1}{2s}[d(fx, Qy) + d(gy, Px)], \frac{d(fx,Px)(1+d(gy,Qy))}{1+d(fx,gy)}, \frac{d(fx,Qy)(1+d(gy,Px))}{1+d(fx,gy)}\right\}\right).$$

If f(X) and g(X) are subsets of X, then we have the following:

1. f and P have a coincidence point,
2. g and Q have a coincidence point,
3. f and P have a common fixed point provided that f and P are weakly compatible at v and ffv = fv for any v $\in$ C(f, P),
4. g and Q have a common fixed point provided that g and Q are weakly compatible at v and ggv = gv for any v $\in$ C(g, Q),
5. f, g, P,Q have a common point provided that both (3) and (4) are true.

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