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Fractional free electron laser equation and generalized Mittag-Leffler function

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Abstract

In this decade fractional free electron laser (FEL) equation are studied due to their utility and importance in mathematical physics, The aim of present paper is to find the solution of generalized fractional order free electron laser (FEL) equation, using Generalized Mittag-Leffler function which is recently given by Salim and Faraj [13]. The results obtained here is moderately universal in nature. Special cases, relating to the exponential function is also considered.

Keywords: Fractional free electron laser (FEL) equation, Generalized Mittag-Leffler function, Riemann-Liouville operator.

Mathematics Subject Classification: 33C60, 33E12, 82C31, 26A33.

1. Introduction:

The Fractional Free Electron Laser Equation:

The unsaturated behaviour of the free electron laser (FEL) is governed by the following first order integro differential equation of Volterra – type [3,4]. :

$$D_T^\alpha a(T) = -i\pi g_0 \int_0^T \xi a(T-\xi) e^{iv\xi} d\xi, \quad 0 \leq T < 1 \quad (1.1)$$

where T is a dimensionless time variable, g_0 is a positive constant known as the small-signal gain and the constant v is the detuning parameter. The functional $a(T)$ is a complex-field amplitude which is assumed to be dimensionless and satisfies the initial condition $a(0) = 1$. Here we employ the Riemann-Liouville definition of fractional integral equation defined by a simplified version of (1.1) changing the scale by putting $t = X\sigma$ and $a = 0$ this yields

$$R_x^\alpha f(x) \equiv I_x^\alpha f(x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} f(x\sigma) d\sigma, \quad \text{Re } \sigma \geq 0 \quad (1.2)$$

The definition (1.2) can be written as

$$R_x^\alpha f(x) \equiv I_x^\alpha f(x) = \frac{d^{-n}}{dx^{-n}} R_x^{\alpha+n} f(x), \quad \text{Re } (\alpha + n) > 0 \quad (1.3)$$

Boyadjiev et al. [3] have treated a non-homogeneous case of (1.2) in which the ordinary first derivative D_T is replaced by the fractional D_T^α with $\alpha > 0$, that is

$$D_T^\alpha a(T) = \lambda \int_0^T t a(T-t) e^{ivt} dt + \beta e^{ivt}, \quad 0 \leq T \leq 1 \quad (1.4)$$

with $\beta, \lambda \in \mathbb{C}$ and $v \in \mathbb{R}$. Furthermore the following generalization of (1.4) has been considered by the authors [2]

$$D_T^\alpha a(T) = \lambda \int_0^T t^\delta a(T-t) e^{ivt} dt + \beta e^{ivt}, \quad 0 \leq T \leq 1 \quad (1.5)$$

where $\beta, \lambda \in \mathbb{C}$, $v \in \mathbb{R}$ and $\delta > -1$. In the present section, we investigate a further generalization of equation (1.5), whereby the exponential term is replaced by the M-SERIES

1.1 Generalized Mittag-Leffler function:

Here, first the notation and the definition of the Generalized Mittag-Leffler function, introduced by Ahmad Faraj, Tariq Salim [13] has been given as

$$E_{\alpha, \beta, n}^{a_1, b_1, m}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km}}{(b_1)_{kn}} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (2.1)$$

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Here $\alpha, \beta \in C, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0$, $(a_1)_{km}, (b_1)_{kn}$ are the pochhammer symbols and m, n are non-negative real numbers.

2. The Generalized Equation:

The generalization of equation (1.5) obtained by replacing $E_{\alpha, \beta, n}^{a_1, b_1, m}(z)$ by $E_{\alpha, \beta, n}^{a_1, b_1, m}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km}}{(b_1)_{kn}} \frac{z^k}{\Gamma(\alpha k + \beta)}$ takes the form

$$D_T^\alpha a(T) = \lambda \int_0^T t^\delta a(T-t) E_{\alpha, \beta, n}^{a_1, b_1, m}(ivt) dt + \beta E_{\alpha, \beta, n}^{a_1, b_1, m}(ivT) \tag{3.1}$$

$$0 \leq T \leq 1$$

where $\alpha, \beta, \lambda \in C; v \in R, \alpha > 0, \delta > -1$, Given real numbers b_k , the corresponding initial conditions are :

$$D_T^{\alpha-k} a(T) \Big|_{T=0} = b_k, \quad k = 1, 2, 3, \dots, N \tag{3.2}$$

with $N = [\alpha] + 1$, so that $N - 1 \leq \alpha < N$.

Equation (1.5) can be deduced from (3.1) by taking $\alpha = 1$ while (3.4) follows when $\alpha = 1, \delta = 1$. To obtain the solution of (3.1) for the given initial conditions (3.2), we use (3.2).

Let $\xi = T - t$ in (3.1), so that

$$D_T^\alpha a(T) = \lambda \int_0^T (T-\xi)^\delta a(\xi) E_{\alpha, \beta, n}^{a_1, b_1, m}(iv(T-\xi)) d\xi + \beta E_{\alpha, \beta, n}^{a_1, b_1, m}(ivT) \tag{3.3}$$

Using the series representation for $E_{\alpha, \beta, n}^{a_1, b_1, m}(z)$ we get

$$a(T) = a_0(T) + \lambda I_T^\alpha \sum_{k=0}^{\infty} \frac{(a_1)_{km}}{(b_1)_{kn}} \frac{(iv)^k}{\Gamma(\alpha k + \beta)} \int_0^T (T-\xi)^{\delta+k} a(\xi) d\xi + \beta I_T^\alpha \left[\sum_{k=0}^{\infty} \frac{(a_1)_{km}}{(b_1)_{kn}} \frac{(iv)^k}{\Gamma(\alpha k + \beta)} \right] \tag{3.4}$$

Where $a_0(T) = \sum_{k=1}^N \frac{b_k}{\Gamma(\alpha - k + 1)} T^{N-k}$ (3.5)

$$\int_0^T \int_0^t u(T, s) ds dT = \int_0^t \int_s^T u(T, s) dT ds \tag{3.6}$$

We obtain the following result

$$a(T) = a_0(T) + \frac{\lambda}{\Gamma(\alpha + 1)} \int_0^T a(\xi) (T-\xi)^{\alpha+\delta} E_{\alpha, \beta, n}^{a_1, b_1, m}(iv(T-\xi)) d\xi + \frac{\beta \Gamma(k+1) T^\alpha}{\Gamma(\alpha + k + 1)} E_{\alpha, \beta, n}^{a_1, b_1, m}(ivT) \tag{3.7}$$

Since (37) is a Volterra integral equation with continuous kernel, it admits a unique continuous solution (2.4). Finally, we consider some special cases of the generalized fractional integro-differential equation of volterra – type (3.1)

1. If $\alpha = 1, \beta = 1$ and there is no upper and lower parameter

$$D_T^\alpha a(T) = \lambda \int_0^T t^\delta a(T-t) e^{iv(T-t)} d\xi + \frac{\beta \Gamma(k+1) T^\alpha e^{ivT}}{\Gamma(\alpha + k + 1)} \tag{3.8}$$

Equivalently

$$D_T^\alpha a(T) = \lambda \int_0^T t^\delta a(T-t) {}_0F_0(-; -; iv(T-\xi)) + \beta \frac{\Gamma(k+1)}{\Gamma(\alpha + K + 1)} T^\alpha {}_0F_0(-; -; ivT) \tag{2.9}$$

3. Conclusion:

In this present work, we have introduced a fractional generalization of the standard free electron laser (FEL) equation. The results of the Advanced generalized fractional free electron laser (FEL) equation and its special case are same as the results of Al-Shammery, A, Kalla, and Khajah, [2](2003).

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