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Vector convex functions for differentiable multiobjective fractional programming problem

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Abstract
The concept of Differentiable multiobjective fractional programming problem under V-convex functions and some other classes of non linear Composite problems as a special case.

Keywords: V-convex functions, multiobjective fractional programming

1. Introduction

In the differentiable case, Jeyakumar and Mond [6] defined a vector invexity that avoids the major difficulty of verifying that the inequality holds for the same function η for invex functions Jeyakumar and Mond [6] established sufficient optimality criteria under V-pseudo invexity and obtained duality results under these assumptions. Egudo & Hanson [3] used the concept of Zhao [7] to generalize, the concept of V-invexity of Jeyakumar and Mond [6] to the non smooth case by replacing the gradients with gradients of Clarke [2] V. Jeyakumar and Mond [6] defined on generalized convex mathematical programming. But they not consider this in multiobjective fractional programming problem. Hence, in this paper an attempt is mode to fill the gap in the aim of research we extended the concept of Differentiable multiobjective fractional programming problem under Vector convex functions and some other classes of non linear Composite problems as a special case.

2. Notations & Preliminaries

Consider constrained multiobjective fractional optimization problem.

(VFP) V- Minimize $\left[\frac{f_i(x)}{g_i(x)}, ----, \frac{f_p(x)}{g_p(x)}\right]$

Subject to $h_j(x) \le 0,\ j = 1,2.......,m$

Where $\frac{f_i}{g_i} : X_\circ \to R$ and $h_j : X_\circ \to R^m$ are differential functions and $x_\circ$ is an open set in R^n. Here the symbol V-minimize stands for vector minimization. This is the problem of finding the set of weak minimum for points (VFP). When P=1, the problem (VPF) reduces to a Scalar optimization problem and it is denoted by (FP). Convexity of the Scalar problem (FP) is characterized by the inequalities.

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(a)}{g_i(a)} - \frac{f_i^1(a)}{g_i^1(a)}(x-a) \ge 0$$

$$h_j(x) - h_j(a) - h_j^1(x)(x-a) \ge 0 \forall x, a \in x_\circ.$$

The functional form $(x-a)$ here plays no role in establishing the following two well-known properties in scalar convex programming:

(s). Every feasible Kuhn - Tucker point is a global minimum
(w) weak duality holds between (FP) and its associated dual problem. Having this in mind, considered problem (FP) for which there exists a function $\eta : X_\circ * X_\circ \to R^n$ such that

(I) $\frac{f_i(x)}{g_i(x)} - \frac{f_i(a)}{g_i(a)} - \frac{f_i^1(a)}{g_i^1(a)}\eta(x,a) \ge 0$

$$h_j(x) - h_j(a) - h_j^1(a)\mu(x,a) \ge 0, \forall x, a \in x_\circ.$$

and showed that such problems (known now as invex problem) also. Possess properties (s) and (w). Since then, various generalization of conditions (I) to multiobjective problems and many properties of functions that satisfy (I) have been established in the literature. However, the major difficulty is that the invex problems require the same function $\eta(x,a)$ for the objective function and the constraints. This requirement turns out to be a severe restriction in applications. Because of this restriction, pseudo liner multiobjective, problems and certain non-linear multiobjective fractional programming problems require separate treatment as far as optimality and duality properties are concerned. In this chapter we show how this situation can be improved and how the properties (s) and (w) can be extended to hold for generalized convex multiobjective problems and certain multi- objective fraction problems. To this, we modify the condition (I) in the next section as follows:

3. New classes & generalized convex vector functions

A vector function $\frac{f_i}{g_i} : X_\circ \to R^p$ is said to be V-Convex if there exist functions $\mu : X_\circ * X_\circ \to R^p$ and $\alpha_i : X_\circ * X_\circ \to R_+ \backslash^{\{0\}}$ such that for each $x, a \in X_\circ$, and for i=1,2,……,p,

$$\frac{f_i(x)}{g_i(x)}-\frac{f_i(a)}{g_i(a)}-\alpha(x,a)\frac{f_i^1(a)}{g_i^1(a)}(x-a)\geq 0,$$

When p=1, the definition of Vector Convexity reduces to the notion of convexity. The problem of (VPF) is said to be V-Convex if the vector function

$$\left(\frac{f_1}{g_2},\frac{f_2}{g_2}......\frac{f_p}{g_p}\right)\&\left(h_1,h_2,.....,h_m\right)$$

is V- convex. Equivalently, Vector Convexity of (VFP) means that there exists function $\mu : X_\circ * X_\circ \to R^n$ and $\alpha_i,\beta_j : X_\circ * X_\circ \to R_+ \setminus \{0\}, \quad i=1,2,\ldots,p, j=1,2,..m$ such that

$$(VI)x,a\in X_\circ \Rightarrow \begin{cases}\frac{f_i(x)}{g_i(x)}-\frac{f_i(a)}{g_i(a)}-\alpha_i(x,a)\frac{f_i^{'}(a)}{g_i^{'}(a)}(x-a)\\ h_j(x)\text{ - }h_j(a)\text{ - }\beta_j(x,a)h_j^{'}(a)(x-a)\end{cases}$$

Let $\alpha_i(x,a)=1=\beta_j(x,a), for\, i=1,2,\ldots p,\ j=1,2,\ldots,m$ and p=1, the scalar problem (FP) becomes the strongly pseudo convex programming problem.

Example 2.2.1

Consider the function $Z:R^n \to R^p$ defined by

$$Z(x)=\left(\frac{f_1}{g_1}(\varphi\varphi(x)),\ldots,\frac{f_p}{g_p}(\varphi\varphi(x)\right)$$

where $\frac{f_i}{g_i}:R^n \to R, i=1,2,\ldots,p$ are strongly pseudo-convex functions with real positive functions $\alpha_i(x,a),\varphi:R^n\to R^n$ is subjective with $\phi^{'}(a)$ onto for each $a\in R^n$. Then, the function Z is V-convex. To see this, Let $x,a\in R^n$

$u=\phi(x), v=\phi(a),$ Then, by strong pseudo convexity,

$$\frac{f_i(\varphi\varphi(x)}{g_i(\varphi\varphi(x)}-\frac{f_i(\varphi\varphi(a)}{g_i(\varphi\varphi(a)}=\frac{f_i(u)}{g_i(u)}-\frac{f_i(v)}{g_i(v)}\geq\alpha_i(u,v)\frac{f_i^{'}(u)}{g_i^{'}(v)}(u-v)$$

Since $\phi^{'}(a)$ is onto $(u-v)=\varphi^{'}(a)(x-a)$ is solvable for some $(x-a)\in R^n$

Hence

$$\frac{f_i(\varphi\varphi(x)}{g_i(\varphi\varphi(x)}-\frac{f_i(\varphi\varphi(a)}{g_i(\varphi\varphi(a)}\geq\alpha_i(u,v)\frac{f_i^{'}(v)}{g_i^{'}(v)}\varphi(a)(x-a)$$

$$=\overline{\alpha_i}(x,a)(f_i.\phi)^{'}(a)(x-a),$$

Where $\overline{\alpha_i}(x,a)=\alpha_i(\phi(x),\phi(a))>0.$

Example 2.2

Consider the composite vector function $Z(X)=\left(\frac{f_i(F_1(x)}{g_i(G_1(x)},\ldots,\frac{f_p(F_p(x)}{g_p(G_p(x)}\right)$, where for each i=1,2,…,p. $\frac{F_i}{G_i}:X_\circ\to R$ is continuously differentiable and pseudo linear with the possible proportional function $\alpha_i(.,.)$ and $\frac{f_i}{g_i}:R\to R$ is convex. The Z is V- Convex with this follows from the following convex inequality and pseudo linear equality conditions.

$$f_i\left(\frac{F_i(x)}{G_i(x)}\right)-f_i\left(\frac{F_i(a)}{G_i(a)}\right)\geq f_i^{'}\left(\frac{F_i(a)}{G_i(a)}\right)\left(\frac{F_i(x)}{G_i(x)}-\frac{F_i(a)}{G_i(a)}\right)$$

$$=f_i^{'}\left(\frac{F_i(a)}{G_i(a)}\right)\alpha_i(x,a)\frac{F_i^{'}(a)}{G_i^{'}(a)}(x-a)$$ for a simple numerical example of a composite vector function, consider

$$Z(x_1,x_2)=\left(e^{x_1/x_2},\frac{x_1-x_2}{x_1+x_2}\right), \text{Where}$$

$$X_\circ=\{(x_1,x_2)\in R^2/x_1\geq 1, x_2\geq 1\}.$$

We now show that the Vector Convexity is preserved under smooth convex transformation.

4. Necessary Theorem

Theorem: Let $\psi:R\to R$ be differentiable and convex with positive derivative everywhere. Let $P:X_\circ\to R^p$ be V-convex. Then, the function

$P_\psi(x)=(\psi(p,(x)),..\psi(h_p(x)), x\in X_\circ$ is V-Convex

Proof: Let $x,a\in X_\circ$, Then from the monotonicity of ψ and Vector Convexity of P, we get

$$\psi(P_i(x))\geq\psi(P_i(a))+\alpha_i(x,a)P_i^{'}(a)(x-a))\geq\psi(P_i(a))+\psi^{'}(P_i(a))\alpha_i(x,a)P^{'}(a)(x-a)$$

$$=\psi(P_i(a))+\alpha_i(x,a)(\psi.P_i)^{'}(a)(x-a)$$

Thus, $P.\psi(x)$ is V-Convex the notations of pseudo-convexity and quasi-convexity for a scalar function can now be extended to a vector function. A vector function $\frac{f_i}{g_i}:X_\circ\to R^p$ is said to be V-pseudo convex if there exists functions $\mu:X_\circ * X_\circ\to R^p$ and $\beta_i:X_\circ * X_\circ$ such that for each $x,a\in X_\circ$,

$$\sum_{i=1}^{p}\frac{fi'(a)}{gi'(a)}(x-a)\geq 0\Rightarrow\sum_{i=1}^{p}\beta_i(x,a)\frac{f_i(x)}{g_i(x)}\geq\sum_{i=1}^{p}\beta_i(x,a)\frac{f_i(a)}{g_i(a)}$$

The vector function $\frac{f_i}{g_i}$ is said to be V-quasi-convex if there exists functions $\mu : X_\circ * X_\circ \to R^p$ and $\bar{r}_i : X_\circ * X_\circ \to R_+ \setminus \{0\}$ such that for each $x, a \in X_\circ$, $\sum_{i=1}^{p} \bar{r}_i(x,a)\frac{f_i(x)}{g_i(x)} \le \sum_{i=1}^{p} \bar{r}_i(x,a)\frac{f_i(a)}{g_i(a)}$ is inconsistent. Assume that $\sum_{i=1}^{p} \tau_i \frac{f_i'(a)}{g_i'(a)} = 0$, for some $0 \ne \tau \in R^p, \tau_i \ge 0$. Suppose that a is not a weak minimum for $\frac{f_i}{g_i}$. Then there exists $x_\circ \in R^n$ such that $\frac{f_i(x_\circ)}{g_i(x_\circ)} \le \frac{f_i(a)}{g_i(a)}$, i=1, 2,…,p. Since $\frac{f_i}{g_i}$ is V-convex, there exist $\alpha_i(x_\circ, a), i = 1,2,..,p$

$$\frac{1}{\alpha_i(x_\circ,a)}\left(\frac{f_i(x_\circ)}{g_i(x_\circ)} - \frac{f_i(a)}{g_i(a)}\right) \ge \frac{f_i'(a)}{g_i'(a)}(x_\circ - a).$$

So, $\sum_{i=1}^{p}\left(\frac{\tau_i}{\alpha_i(x_\circ,a)}\right)\frac{f_i(x_\circ)}{g_i(x_\circ)} - \frac{f_i(a)}{g_i(a)}$ <0,

and hence $\sum_{i=1}^{p} \tau_i \frac{f_i'(a)}{g_i'(a)}(x_\circ - a) < 0.$ This is a contradiction.

5. Sufficiency theorem

Consider the multiobjective fractional problem (VFP). Suppose that the Lagrange multiplier conditions that $\sum_{i=1}^{p}\tau_i \frac{f_i'(a)}{g_i'(a)} + \sum_{j=1}^{m}\lambda_j h_j'(a) = 0, \lambda_j h_j(a) = 0, \tau \in R^p, \tau \ne 0, \tau \ge 0, \tau \in R^m, \lambda \ge 0$, hold at a feasible point $a \in X_\circ$. If $\left(\tau_1\frac{f_1}{g_1}, \tau_2\frac{f_2}{g_2}, ..., \tau_p\frac{f_p}{g_p}\right)$ is V-pseudo convex with respect to μ and $(\lambda_1 h_1, ..., \lambda_m h_m)$ is V-quasi convex with respect to μ then a is a global weak minimum point for (VFP).

Proof:- Suppose that a is not a global minimum point. Then there exists feasible $x_\circ \in X_\circ$ such that $\frac{f_i(x_\circ)}{g_i(x_\circ)} < \frac{f_i(a)}{g_i(a)}$, i=1, 2,.., p. So, $\sum_{i=1}^{p}\beta_i(x_\circ,a)\tau_i\frac{f_i(x_\circ)}{g_i(x_\circ)} < \sum_{i=1}^{p}\beta_i(x_\circ,a)\tau_i\frac{f_i(a)}{g_i(a)}$. Now by the V-pseudo convexity condition, we get $\sum_{i=1}^{p}\tau_i\frac{f_i'(a)}{g_i'(a)}\mu(x_\circ,a) < 0$. Since the Lagrangian condition holds at a, $\sum_{j=1}^{m}\lambda_j h_j'(a)\mu(x_\circ,a) > 0$ From the V-quasi-convexity condition, we get $\sum_{j=1}^{m}\bar{r}_j(x_\circ,a)\lambda_j h_j(x_\circ) > \sum_{j=1}^{m}\bar{r}_j(x_\circ,a)\lambda_j$ this is a contradiction, since $\lambda_j h_j(x_\circ) \le 0, \lambda_j h_j(a) = 0, and\, \bar{r}_j(x_\circ,a) > 0,\, for j = 1,2,...,m$

6. Duality

Dual Formulation

V-maximize $\left(\frac{f_1}{g_1}(\xi), ..., \frac{f_p}{g_p}(\xi)\right)$,

(VFD) Subject to

$$\sum_{i=1}^{p}\tau_i\frac{f_i'}{g_i'}(\xi) + \sum_{j=1}^{m}\lambda_j h_j'(\xi) \ge 0, \lambda_j \ge 0, \tau_i \ge 0, \tau e = 1, \lambda_j h_j(\xi) \ge 0,\ j = 1,2,..,m.$$

Where e= (1, 1… 1). Note that the Lagrangian conditions in sufficiency theorem hold for (VFP) at a weak minimum point a under a constraint qualification and they can equivalently be written as

$$\sum_{i=1}^{p}\tau_i\frac{f_i'(a)}{g_i'(a)} + \sum_{j=1}^{m}\lambda_j h_j'(a) = 0, \lambda \ \ge 0, \tau \ \ge 0, \tau e = 1, \lambda_j h_j(a) = 0,\ j = 1,2,..,\ m.$$

Weak Duality Theorem: Consider the multiobjective fractional problem (VFP) and (VFD). Let x feasible for (VFP) and (ξ, τ, λ) be feasible for (VFD). If $\left(\tau_1\frac{f_1}{g_1}, \tau_2\frac{f_2}{g_2}, ..., \tau_p\frac{f_p}{g_p}\right)$ is V-pseudo convex and $(\lambda_1 h_1, ..., \lambda_m h_m)$ is V-quasi convex with respect to the same μ, then

$$\left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right)^T - \left(\frac{f_1(\xi\xi}{g_1(\xi\xi}, ..., \frac{f_p(\xi\xi}{g_p(\xi\xi}\right)^T \notin \text{in} + R^p{}_+.$$

Proof: from the feasibility conditions, $\lambda_j h_j(x) - h_j h_j(\xi\xi \le 0$, for each j=1,2,..,m. Since $\bar{r}_j(x,\xi)$ is positive $\sum_{j=1}^{m}\bar{r}_j(x,\xi)[h_j(x) - h_j(\xi\xi] \le 0$. Hence, $\sum_{j=1}^{m}\lambda_j h_j'(\xi\xi\mu(x,\xi) \le 0,$ and so, $\sum_{i=1}^{p}\tau_i\frac{f_i'(\xi\xi}{g_i'(\xi\xi}\mu(x,\xi) \ge 0,$ the conclusion now follows from the V-pseudo convexity condition since $\tau e = 1$ and $\beta_i(x,\xi) > 0$.

Theorem: Strong Duality Theorem

Assume that a is a weak minimum of (VFP) and that a suitable constraint qualification is satisfied at a. Then there exist (τ, λ) such that (a, τ, λ) is a feasible for (FVD).

Proof: since a is a weak minimum for (FVP) and a constraint qualification is satisfied at a, from the Lagrangian conditions, there exists (τ, λ) such that (a, τ, λ) is a feasible for (VFD). Clearly the values of (VFP) and (VFD) are equal at a, since the objective function for both problems are the same. By the generalized Vector Convexity hypothesis, weak duality holds, hence if (a, τ, λ) is not a weak optimum for

(VFD), there must exist (ξ, τ^*, λ^*) feasible for (VFD), $\xi \neq a$ such that

$$\left(\frac{f_1(\xi)}{g_1(\xi)},\ldots.,\frac{f_p(\xi)}{g_p(\xi)}\right)^T - \left(\frac{f_1(a)}{g_1(a)},\ldots.,\frac{f_p(a)}{g_p(a)}\right)^T \in \text{in} + R^p{}_+$$

Contradicting weak duality.

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