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Construction of family of triples where the sum of any two members of a triple is a perfect square

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Abstract
This paper deals with the construction of families of integer triples where, in each triple, the sum of any two members is a perfect square.

Keywords: integer pairs, integer triples

1. Introduction

Every advanced under graduate and graduate student of Mathematics as well as any researcher in Number Theory is familiar with Pythagorean triple which provides the relation between three sides of a right-angled triangle in addition to the concept of three integers representing an arithmetic progression, geometric progression and harmonic progression respectively. In this context, one may refer [1] wherein the authors have given a collection of problems with solutions on integer triples in arithmetic progression.

Similar to a Pythagorean triple, we have a triple known as Heronian triple defined as follows: If a, b, c represent the sides of a triangle with integer area, then the triple (a, b, c) is known as Heronian triple. It is worth to note that not every Heronian triple is a Pythagorean triple. For example: (4, 13, 15) is a Heronian triple but not Pythagorean triple whereas (5,12,13) is both Heronian triple as well as Pythagorean triple. Also, we have a triple known as Eisenstein triple which is a set of integers which are the lengths of the sides of a triangle where one of the angle is 60^0. In other words, An Eisenstein triple (a, b, c) consists of three positive integers $a < c < b$ such that $a^2 - ab + b^2 = c^2$

No doubt that the triples in integers may be formulated in varieties of ways. For a review of various problems on triples, one may refer [2-6]. It is therefore towards this end, we are motivated to search for families of triples where, in each triple, the sum of any two of its members is a perfect square.

Construction of triples

Let r, s be two non-zero distinct positive integers such that $r > s > 0$
Write

$$a = 4r^2s^2, \quad b = (r^2 - s^2)^2, \quad r > s > 0$$

$$\Rightarrow \quad a + b = (r^2 + s^2)^2$$, a perfect square

Let c be any non-zero integer distinct from a, b such that

$$a + c = \beta^2 \tag{1}$$

$$b + c = \gamma^2 \tag{2}$$

Solving (1) and (2) $\Rightarrow \quad a - b = \beta^2 - \gamma^2$

Employing the identity

$$(A+1)^2 - A^2 = 2A + 1$$

and performing a few calculations, we have

$$c = A^2 - b = \left[3r^2s^2 - \frac{1}{2}\left(r^4 + s^4 + 1\right)\right]^2 - \left(r^2 - s^2\right)^2$$

For c to be an integer, r and s should be of different parity.

Case (i)

Let $r = 2k\,, \quad s = 2l + 1, \quad k > l > 0$

Therefore, $a = 16k^2\,(2l+1)^2$

$$b = \left(4k^2 - (2l+1)^2\right)^2$$

$$c = \left(12k^2(2l+1)^2 - \frac{1}{2}\left[16k^4 + (2l+1)^4 + 1\right]\right)^2 - \left[4k^2 - (2l+1)^2\right]^2$$

Hence, (a,b,c) is the required triple where the sum of any two of them is a perfect square.

Table 1: Numerical Examples

k	l	a	b	c	$a+c$	$b+c$	$a+b$
2	1	576	49	69120	264^2	263^2	25^2
3	2	3600	121	3024000	1740^2	1739^2	61^2
4	2	6400	1521	5947200	2440^2	2439^2	89^2

Case (ii)

Let $r = 2k + 1\,, \quad s = 2l\,, \quad k \ge l > 0$

Here, $a = 16l^2\,(2k+1)^2$

$$b = \left((2k+1)^2 - 4l^2\right)^2$$

$$c = \left(12l^2(2k+1)^2 - \frac{1}{2}\left[(2k+1)^4 + 16l^4 + 1\right]\right)^2 - \left[(2k+1)^2 - 4l^2\right]^2$$

Hence, (a,b,c) is the required triple where the sum of any two of them is a perfect square.

Table 2: Numerical Examples

k	l	a	b	c	$a+b$	$b+c$	$a+c$
5	2	7744	11025	2681856	137^2	1641^2	1640^2
4	4	20736	289	104509440	145^2	10223^2	10224^2
3	6	28224	9025	92131776	193^2	9599^2	9600^2

Construction of various other families of triples where the sum of any two members of a triple is a perfect square is illustrated below:

I. Consider the non-homogeneous ternary cubic equation given by

$$x^3 + y^3 = z^2 \tag{1.1}$$

Observe that (1.1) is satisfied by

$$x = m\left(m^3+n^3\right),\quad y = n\left(m^3+n^3\right),\quad z = \left(m^3+n^3\right)^2 \tag{1.2}$$

We present below different families of triples satisfying the required conditions.

Triple 1
Assume

$$x = 2p\,,\quad y = 2q+1 \qquad (1.3),\ (1.4)$$

From (1.2), (1.3) and (1.4), we have

$$p = \frac{1}{2}\left[m\left(m^3+n^3\right)\right],\quad q = \frac{1}{2}\left[n\left(m^3+n^3\right)-1\right] \tag{1.5}$$

The values of p and q are integers when

$$m = 2r\,,\quad n = 2s+1\,,\quad r, s \in Z - \{0\} \tag{1.6}$$

Substituting (1.6) in (1.5), we have

$$p = 8r^4 + r + 8rs^3 + 12rs^2 + 6rs = f_1(r,s)$$

$$q = 4r^3 + 8s^4 + 16s^3 + 12s^2 + 4s + 8r^3 s = g_1(r,s)$$

Let $a_1(r,s) = 8 f_1^3(r,s),\quad b_1(r,s) = \left(2g_1(r,s)+1\right)^3$

$\Rightarrow\quad a_1(r,s) + b_1(r,s)$ is a perfect square.

Let $c_1(r,s)$ be any non-zero integer distinct from $a_1(r,s), b_1(r,s)$ such that

$$a_1(r,s) + c_1(r,s) = \alpha^2 \tag{1.7}$$

$$b_1(r,s) + c_1(r,s) = \beta^2 \tag{1.8}$$

Subtraction of (1.8) from (1.7) gives

$$\alpha^2 - \beta^2 = a_1(r,s) - b_1(r,s) \tag{1.9}$$

which is satisfied by $\alpha = A+1,\quad \beta = A$, $A = \frac{1}{2}\left[a_1(r,s) - b_1(r,s) - 1\right]$

(1.10) From (1.10) and (1.8), we have

$$c_1(r,s) = \frac{1}{4}\left(\left[a_1(r,s) - b_1(r,s) - 1\right]^2 - 4b_1(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $\left(a_1(r,s), b_1(r,s), c_1(r,s)\right)$ is the required triple where the sum of any two of them is a perfect square.

Triple 2
Assume

$$x = 2p + 2, \quad y = 2q - 1 \tag{1.11), (1.12}$$

From (1.2), (1.11) and (1.12), we have

$$p = \frac{1}{2}\left[m\left(m^3 + n^3\right) - 2\right], \quad q = \frac{1}{2}\left[n\left(m^3 + n^3\right) + 1\right] \tag{1.13}$$

The values of p and q are integers when

$$m = 2r + 2, \quad n = 2s + 1, \quad r, s \in Z - \{0\} \tag{1.14}$$

Substituting (1.14) in (1.13), we have

$$p = 8r^4 + 32r^3 + 48r^2 + 33r + 8s^3 + 12s^2 + 6s + 8s^3 r + 12s^2 r + 6rs + 8 = f_2(r, s)$$

$$q = 4r^3 + 12r^2 + 12r + 8s^4 + 16s^3 + 12s^2 + 12s + 8r^3 s + 24r^2 s + 24rs + 5 = g_2(r, s)$$

Let $a_2(r,s) = (2 f_2(r,s) + 2)^3$, $\quad b_2(r,s) = (2g_2(r,s) - 1)^3$

$\Rightarrow \quad a_2(r,s) + b_2(r,s)$ is a perfect square.

Let $c_2(r,s)$ be any non-zero integer distinct from $a_2(r,s), b_2(r,s)$ such that

$$a_2(r,s) + c_2(r,s) = \alpha^2$$

$$b_2(r,s) + c_2(r,s) = \beta^2$$

Following the procedure as in Triple: 1, it is seen that

$$c_2(r,s) = \frac{1}{4}\left(\left[a_2(r,s) - b_2(r,s) - 1\right]^2 - 4b_2(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $(a_2(r,s), b_2(r,s), c_2(r,s))$ is the required triple such that the sum of any two of them is a perfect square.

Triple 3
Assume

$$x = 6p + 6, \quad y = 4q + 1 \tag{1.15), (1.16}$$

From (1.2), (1.15) and (1.16), we have

$$p = \frac{1}{6}\left[m\left(m^3 + n^3\right) - 6\right], \quad q = \frac{1}{4}\left[n\left(m^3 + n^3\right) - 1\right] \tag{1.17}$$

The values of p and q are integers when

$$m=6(r+1),\quad n=4s+2t+1,\quad r,s,t\in Z-\{0\} \tag{1.18}$$

Substituting (1.18) in (1.17), we have

$$p=r\left(216r^3+864r^2+1296r+865\right)+4s\left(16s^2+12s+3\right)+2t\left(4t^2+6t+3\right)$$
$$+4rs\left(16s^2+12s+3\right)+2rt\left(4t^2+6t+3\right)+48st\left(1+2s+t\right)\left(1+r\right)+216=f_3(r,s)$$

$$q=6r\left(9r^2+27r+27\right)+4s\left(16s^3+16s^2+6s+55\right)+2t\left(2t^3+4t^2+3t+55\right)$$
$$+8st\left(4t^2+16s^2+6t+12s+12st+3\right)+108r\left(t+2s\right)\left(r^2+3r+3\right)+54=g_3(r,s)$$

Let $a_3(r,s)=216\left(f_3(r,s)+1\right)^3,\quad b_3(r,s)=\left(4g_3(r,s)+1\right)^3$

$\Rightarrow\quad a_3(r,s)+b_3(r,s)$ is a perfect square.

Let $c_3(r,s)$ be any non-zero integer distinct from $a_3(r,s), b_3(r,s)$ such that

$$a_3(r,s)+c_3(r,s)=\alpha^2$$

$$b_3(r,s)+c_3(r,s)=\beta^2$$

After performing a few calculations, it is seen that

$$c_3(r,s)=\frac{1}{4}\left(\left[a_3(r,s)-b_3(r,s)-1\right]^2-4b_3(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $\left(a_3(r,s), b_3(r,s), c_3(r,s)\right)$ is the required triple such that the sum of any two of them is a perfect square.

Triple 4

Assume

$$x=6p-3,\quad y=4q+2 \tag{1.19), (1.20}$$

From (1.2), (1.19) and (1.20), we have

$$p=\frac{1}{6}\left[m\left(m^3+n^3\right)+3\right],\quad q=\frac{1}{4}\left[n\left(m^3+n^3\right)-2\right] \tag{1.21}$$

The values of p and q are integers when

$$m=6r+3,\quad n=4s+6,\quad r,s\in Z-\{0\} \tag{1.22}$$

Substituting (1.22) in (1.21), we have

$$p=216r^4+432r^3+324r^2+324r+32s^3+144s^2+216s+64s^3r$$
$$+288s^2r+432rs+122=f_4(r,s)$$

$$q = 324r^3 + 486r^2 + 243r + 64s^4 + 384s^3 + 864s^2 + 891s + 216r^3 s + 324r^2 s + 162rs + 364 = g_4(r,s)$$

Let $a_4(r,s) = (6 f_4(r,s) - 3)^3$, $b_4(r,s) = (4 g_4(r,s) + 2)^3$

$\Rightarrow$ $a_4(r,s) + b_4(r,s)$ is a perfect square.

Let $c_4(r,s)$ be any non-zero integer distinct from $a_4(r,s), b_4(r,s)$ such that

$$a_4(r,s) + c_4(r,s) = \alpha^2$$

$$b_4(r,s) + c_4(r,s) = \beta^2$$

After some algebra, it is seen that

$$c_4(r,s) = \frac{1}{4}\left([a_4(r,s) - b_4(r,s) - 1]^2 - 4b_4(r,s) \right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $(a_4(r,s), b_4(r,s), c_4(r,s))$ is the required triple such that the sum of any two of them is a perfect square.

Triple 5
Assume

$$x = 4p + 1, \quad y = 2q + 2 \tag{1.23), (1.24}$$

From (1.2), (1.23) and (1.24), we have

$$p = \frac{1}{4}\left[m\left(m^3 + n^3\right) - 1\right], \quad q = \frac{1}{2}\left[n\left(m^3 + n^3\right) - 2\right] \tag{1.25}$$

The values of p and q are integers when

$$m = 4r + 1, \quad n = 2s + 2, \quad r,s \in Z - \{0\} \tag{1.26}$$

Substituting (1.26) in (1.25), we have

$$p = r\left(64r^3 + 48r^2 + 12r + 8s^3 + 24s^2 + 24s + 9\right) + 16r^3 + 12r^2 + 3r + 2s^3 + 6s^2 + 6s + 2 = f_5(r,s)$$

$$q = s\left(64r^3 + 48r^2 + 12r + 8s^3 + 24s^2 + 24s + 9\right) + 64r^3 + 48r^2 + 12r + 8s^3 + 24s^2 + 24s + 8 = g_5(r,s)$$

Let $a_5(r,s) = (4 f_5(r,s) + 1)^3$, $b_5(r,s) = (2 g_5(r,s) + 2)^3$

$$\Rightarrow \quad a_5(r,s)+b_5(r,s)$$ is a perfect square.

Let $c_5(r,s)$ be any non-zero integer distinct from $a_5(r,s), b_5(r,s)$ such that

$$a_5(r,s)+c_5(r,s)=\alpha^2$$

$$b_5(r,s)+c_5(r,s)=\beta^2$$

After performing a few calculations, it is seen that

$$c_5(r,s)=\frac{1}{4}\left(\left[a_5(r,s)-b_5(r,s)-1\right]^2-4b_5(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $\left(a_5(r,s), b_5(r,s), c_5(r,s)\right)$ is the required triple such that the sum of any two of them is a perfect square.

Triple 6
Assume

$$x=2p+6\,,\quad y=3q-6 \qquad (1.27), (1.28)$$

From (1.2), (1.27) and (1.28), we have

$$p=\frac{1}{2}\left[m\left(m^3+n^3\right)-6\right],\quad q=\frac{1}{3}\left[n\left(m^3+n^3\right)+6\right] \qquad (1.29)$$

The values of p and q are integers when

$$m=2r+2\,,\quad n=3s+6\,,\quad r,s\in Z-\{0\} \qquad (1.30)$$

Substituting (1.30) in (1.29), we have

$$p=r\left(8r^3+24r^2+24r+27s^3+162s^2+324s+224\right)$$
$$+8r^3+24r^2+24r+27s^3+162s^2+324s+221= f_6(r,s)$$

$$q=s\left(8r^3+24r^2+24r+27s^3+162s^2+324s+224\right)$$
$$+16r^3+48r^2+48r+54s^3+324s^2+648s+450=g_6(r,s)$$

Let $a_6(r,s)=\left(2f_6(r,s)+6\right)^3\,,\quad b_6(r,s)=\left(3g_6(r,s)-6\right)^3$

$$\Rightarrow \quad a_6(r,s)+b_6(r,s)$$ is a perfect square.

Let $c_6(r,s)$ be any non-zero integer distinct from $a_6(r,s), b_6(r,s)$ such that

$$a_6(r,s)+c_6(r,s)=\alpha^2$$

$$b_6(r,s)+c_6(r,s)=\beta^2$$

After some algebra, it is seen that

$$c_6(r,s)=\frac{1}{4}\left(\left[a_6(r,s)-b_6(r,s)-1\right]^2-4b_6(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $(a_6(r,s), b_6(r,s), c_6(r,s))$ is the required triple such that the sum of any two of them is a perfect square.

II. Consider the non-homogeneous ternary quintic equation given by

$$x^5+y^5=z^2 \tag{2.1}$$

Observe that (2.1) is satisfied by

$$x=m\left(m^5+n^5\right),\quad y=n\left(m^5+n^5\right),\quad z=\left(m^5+n^5\right)^3 \tag{2.2}$$

Assume

$$x=6p-3,\quad y=4q+2 \qquad (2.3),\ (2.4)$$

From (2.2), (2.3) and (2.4), we have

$$p=\frac{1}{6}\left[m\left(m^5+n^5\right)+3\right],\quad q=\frac{1}{4}\left[n\left(m^5+n^5\right)-2\right] \tag{2.5}$$

The values of p and q are integers when

$$m=6r+3,\quad n=4s+2,\quad r,s\in Z-\{0\} \tag{2.6}$$

Substituting (2.6) in (2.5), we have

$$\begin{aligned} p &= 7776r^6+23328r^5+29160r^4+19440r^3+7290r^2+1490r \\ &\quad +512s^5+1280s^4+1280s^3+640s^2+160s+1024s^5r+2560s^4r \\ &\quad +2560s^3r+1280s^2r+320sr+138= f_1(r,s) \end{aligned}$$

$$\begin{aligned} q &= 3888r^5+9720r^4+9720r^3+4860r^2+1215r+1024s^6 \\ &\quad +3072s^5+3840s^4+2560s^3+960s^2+435s+7776r^5s+19440r^4s \\ &\quad +19440r^3s+9720r^2s+2430rs+137=g_1(r,s) \end{aligned}$$

Let $a_1(r,s)=(6f_1(r,s)-3)^5,\quad b_1(r,s)=(4g_1(r,s)+2)^5$

$\Rightarrow\ a_1(r,s)+b_1(r,s)$ is a perfect square.

Let $c_1(r,s)$ be any non-zero integer distinct from $a_1(r,s), b_1(r,s)$ such that

$$a_1(r,s)+c_1(r,s)=\alpha^2 \tag{2.7}$$

$$b_1(r,s)+c_1(r,s)=\beta^2 \tag{2.8}$$

Subtraction of (2.8) from (2.7) gives

$$\alpha^2-\beta^2=a_1(r,s)-b_1(r,s) \tag{2.9}$$

which is satisfied by $\alpha = A+1, \quad \beta = A$, $A=\frac{1}{2}\left[a_1(r,s)-b_1(r,s)-1\right]$ (2.10) From (2.10) and (2.8), we have

$$c_1(r,s)=\frac{1}{4}\left(\left[a_1(r,s)-b_1(r,s)-1\right]^2-4b_1(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $(a_1(r,s), b_1(r,s), c_1(r,s))$ is the required triple where the sum of any two of them is a perfect square.

Conclusion

In this paper, we have made an attempt to construct family of triples where the sum of any members of a triple is a perfect square. To conclude, one may search for families of triples with different relations among its members.

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