International Journal of Multidisciplinary Research and Development
Online ISSN: 2349-4182 Print ISSN: 2349-5979
www.allsubjectjournal.com
Volume 2; Issue 12; December 2015; Page No. 258-260

A common fixed point theorem in uniformly convex banach space

Rambyas Patel

Dept. of Mathematics/Computer Science, Sitaram Samarpan Degree College Naraini, Banda (U.P.), 210129, India.

Abstract
In this paper we improve result of Rashwan by removing the condition continuity and replacing the compatibility of mappings of type (A) by weak compatibility.

Keywords: Common fixed point, Banach space, Weak Compatible mappings

1. Introduction

Husain and Sehgal [1] proved common fixed point theorems for a family of mapping. Khan and Imdad [2] extended result of Husain and Sehgal [1] and proved fixed point theorems for a class of mapping. Imdad, Khan and Sessa [4] extended above results and proved common fixed points for three mappings defined on a closed subset of uniformly convex Banach space.

Rashwan [3] extended result of imdad, Khan and sessa [4] by employing four compatible mapping of type (A) instead of weakly commuting mappings and by using one continuous mapping as opposed to two. In the present paper we improve results of Rashwan [3].

Throughout the paper X stands for a Banach space. Let R^+ denote the set of all non-negative real numbers and F be the family of mapping f for s>0, f(s,s,0)<s, f(s,0,λs)<s.

The modulus of convexity of X is a function δ from (0, 2] into (0, 1] define by $\delta(Є)=\inf\{1-{}^{1}/_{2}||x-y||, x, y Є X, ||x||=||y||=1, ||x-y||\geq Є\}$

For our theorem we need the following lemma:

Lemma 1.1[5]: Let X be uniformly convex and Br, the closed ball in X centered at the origin with radius.0. If x_1,x_2,x_3 ЄBr. Satisfy $||x_1-x_2||\geq||x_2-x_3||\geq d>0$ and if $||x_2||\geq(1-{}^{1}/_{2}\delta(d/l))$,

Then $||x_1-x_3||\leq\eta(1-{}^{1}/_{2}\delta(d/l))||x_1-x_2||$.

Now, we begin with some know definition:

Lemma 1.2: For every s>0,γ(s)<s iff $\lim_{n\to\infty}\gamma^n(s)=0$,where γ^n denotes the composition of γ n-time with itself.

Definition 1.1[6]: Let S and T be self mapping on X. Then {S, T} is called a weakly commuting pair on X.

If || STx - TSx ||≤|| Sx – Tx || for all xЄX.

Definition 1.2[7]: Let S,T :X→X be mappings. S and T are said to be compatible if $\lim_{n\to\infty}||STx_n-TSx_n||=0$,

Whenever {xn} is a sequence in X such that

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=s \text{ for some } sЄX.$$

Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible. On the other hand, example are given by Jungck [7] and sessa [6] to show neither of the above implications are reversible.

Definition 13(8): Two self-mappings S and T are said to be weakly compatible if they commute at their coincidence point, i.e. If Tu=Su for some uЄX. Then TSu=STu.

2. Common fixed point theorem

In a recent paper Imdad, KHAN AND sessa [4] proved the following theorem.

Theorem A: Let X be uniformly convex and K a non-empty closed subset of X. Let A, B, S and T be four self-mappings of K satisfying the following condition:

(i) S and T are continuous, AK ⊂ SK ∩ TK.

(ii) {A, S} and {A, T} are weakly commuting pairs on K,

(iii) There exist a function fЄF such thst for all x,yЄk

||Ax-Ay||≤f(||Sx-Ty||,||Sx-Ax||,||Sx-Ay||,||Ty-Ax||,||Ty-Ay||)

Where f has the additional requirements,

(iv) for s>0, f(s,s,0,λs,s)≤βs and f(s,s,λs,0,s)≤βs being β<1 for λ<2 and β=1 For λ=2, λ,β R^+,

(v) f(s,0,s,s,0)<s for s>0. Then there exist a point u in K such thst

a) u is the unique common fixed point of A, S and T.

b) For any x_0ЄK, the Sequence {Axn} defined by $Tx_{2n}=Ax_{2n-1}$; $Sx_{2n+1}=Ax_{2n}$ for n=0, 1, 2, 3…….. converges strongly to u.

Rashwan [2] extended theorem. A for compatible mapping of type (A).

(vi) One of Ak, BK, Sk or TK is co0mplete subspace of X, Then
a) A and S have a coincidence point,
b) B and T have a coincidence point.
(vii) The Pairs {A,S}and {B,T} are weakly compatible, Then
a) A,B,S and T have a common fixed point z in K.
Further Z is the unique common fixed point of A and S and of B and T.

3. Main Result

Theorem 3.1: Let X be uniformly convex and K a non-empty closed subset of X. Let A, B, S and T be mappings on k satisfying the following condition:
(1)Ak ⊂ TK and BK ⊂ SK,
(2) there exist a function f∈F such that for every x,y∈k:
$\|Ax-By\|^2 \le f(\|Sx-Ty\|^2, \|Sx-By\|.\|Tx-Ax\|, \|Sx-Ax\|.\|Ty-Ax\|)$
Where f satisfies for s>0, f(s,s,0)<s, f(s,0,λs)<s
(3) One of AK,SK or TK is complete subspace of X, then
(i) A and S have a coincidence point,
(ii) B and T have a coincidence point.
(4) the pair {A,S} and {B,T} are weakly compatible, then
(iii) A,B,S and T have a common fixed point z in K.
Further z is the unique common fixed point of A and S and of B and T.
Proof : Let x_0∈K, since AK ⊂ TK, BK ⊂ SK, we can always define a sequence {yn}such that

$y2n=Sx2n=Bx2n_{-1},$
$y2n_{+1}=Tx2n_{+1}=Ax2n$, n=1,2,3,....

Let $dn=\|yn-yn_{+1}\|$, n=0,1,2,……..
d = lim n→∞dn.
Now, for an even n, we have
(5) $dn=\|yn-yn_{+1}\|=\| Axn-Bxn_{-1}\|$
$\le f(\|Sxn-Txn_{-1}\|^2, \| Sxn-Bxn_{-1}\|, \| Txn_{-1}-Axn\|, \| Sxn-Axn \|,$
$\| Txn-Axn\|)$
$\le f(\|yn-yn_{-1}\|^2, \| yn-yn\| \| yn_{-1}-yn_{+1}\|, \| yn-yn_{+1}\|$
$\| yn_{-1}-yn_{+1}\|)$
$\le f(\|yn-yn_{-1}\|^2, 0, \| yn-yn_{+1}\|(\| yn_{-1}- yn+yn-yn_{+1}\|))$
$\le f(\|yn-yn_{-1}\|^2, 0, \| yn-yn_{+1} \|(\| yn_{-1}- yn\|+\|yn-yn_{+1}\|))$
$\le f(\|yn-yn_{-1}\|^2, 0, \| yn-yn_{+1} \|\|\| yn- yn_{-1}\|+\|yn-yn_{+1}\|\|\|yn-yn_{+1}\|)$
$\le f(d^2n_{-1}, 0, dn(dn_{-1}+ dn))$

Similarly for an odd n, we obtain (6)
$dn=\|yn-yn_{-1}\|=\| Axn_{-1}-Bxn\|$
$\| Axn_{-1}-Bxn\|^2 \le f(\|Sxn_{-1}-Txn\|^2, \| Sxn_{-1}-Bxn\|.\| Txn-Axn_{-1}\|,$
$\| Sxn_{-1}-Axn_{-1}\|.\| Txn-Axn_{-1}\|)$
$\| Axn_{-1}-Bxn\|^2 \le f(\|yn_{-1}-yn\|^2, \|yn_{-1}-yn_{+1}\|.\| yn-yn \|,$
$\| yn_{-1}-yn\|.\| yn-yn \|)$
$\|yn-yn_{+1}\|^2 \le f(\|yn_{-1}-yn\|, 0, 0)$
$\le f(dn_{-1}, 0, 0)$ [by lemma 1.2]
If $dn> dn_{-1}$, for some n≥1, then $dn_{-1}+dn=\lambda dn$ with λ<2, λ∈R.
Since f is non decreasing in each coordinate variable
dn≤ {f(dn,0,λdn),if n is even} or {f(dn,0,0),if n is odd}
In both cases by (iv) we get dn<βdn< dn for some β<1, βR^+, a contradiction. This means that d=0 as n→∞.
Now, we wish to prove that {yn} is a Cauchy sequence.
Since lim n→∞dn=0. It is sufficient to show that the sequence {y2n} is a Cauchy sequence. If not there is an Є>0 such that for every even integer 2k,k=0,1,2,3…, there exist two sequence {2n(k)}, {2m(k)} with 2k≤2n(k)≤2m(k) for which (7)

$\| yn(k)-ym(k)\|>Є.$

For each even integer 2k, let 2m(k) be the least even integer exceeding n(k) and satisfying (7). Then $\| y2n(k)-y2m(k)_{-2}\| \le Є$ and $\| y2n(k)-y2m(k)\|>Є$.
For each k=0,1,2……., we have $Є< \| y2n(k)-y2m(k)\|$
$\le \| y2n(k)-y2m(k)\|+ \| y2m(k)_{-2}-y2m(k)_{-1}\|+ \| y2n(k)_{-1}-y2m(k)\|$
$\le Є+ d2m(k)_{-2}-d2m(k)_{-1},$

Which implies
(8) lim n→∞$\| y2n(k)-y2m(k)\|=Є$.
Hence for k→∞, we fined that
(9) $\| y2n(k)-y2m(k)_{-1}\| \to Є$ and $\| y2n(k)_{+1}-y2m(k)_{-1}\| \to Є$. Using (2), we deduce that

(10) $\| y2n(k)-y2m(k)\| \leq d2n(k)+\| y2n(k)_{+1}-y2m(k)\|$

$\leq d2n(k)+f(\| y2m(k)_{-1}-y2n(k)\|, d2n(k), \| y2m(k)_{-1}-y2n(k)_{+1}\|,$

$\| y2n(k)-y2m(k)\|, d2n(k))$ by (8), (9),

The upper-semi continuity and non-decreasing properties off, and condition (v), we have from (10) for $k\to\infty$.which is a contradiction.

Therefore {y2n} is a Cauchy sequence in k and so is {yn}. But is a closed subset of a Banach space X, therefore {yn} converges to a point z in k.

Now suppose that SK is complete. Let $u=S^{-1}z$. Then Su=z, By [2],

We have $\| Au-Bx2n_{+1}\|^2 \leq f(\|Su-Tx2n_{+1}\|^2, \| Su-Bx2n_{+1}\|$

$\| Tx2n_{+1}-Au\|, \| Su-Au \|\| Tx2n_{+1}-Au\|)$.

Taking the limit $n\to\infty$, we have

$\| Au-z\|^2 \leq f(\|z-z\|^2, \| z-z\|\| z-Au\|, \| z-Au \|\| z-Au\|)$.

$\leq f(0,0,\| z-Au \|^2)$. By lemma (1.2),

We have $\| Au-z\|^2 \leq (\| z-Az\|^2) < \| z-Au\|^2$.

Which is a contraction, therefore Au=z. thus Su=Au=z. i.e., u is a coincidence point of A and S.

Let $v\in T^{-1}z$, Then Tv=z. By (2), we have $\| Ax2n-Bv\|^2$

$\leq f(\| Sx2n-Tv\|^2, \| Sx2n-Bv\|\| Tv-Ax2n\|, \| Sx2n-Ax2n\|\| Tv-Ax2n\|)$

Taking the limit $n\to\infty$, we have

$\| z-Bv \|^2 \leq f(\|z-z\|^2, \| z-Bv\|\| z-z\|, \| z-z \|\| z-z\|)$

$\| z-Bv\|^2 \leq 0$.

Let $\| z-Bv \| > 0$. Being f non-decreasing in each coordinate variable from (2). We have

$\| z-Bv \| \leq f(\|z-Bv\|, \| z-Bv\|, \| z-Bv\|, 0, \| z-Tv\|)$

Where $\lambda \leq 1 \leq 2$. Applying (iv) then we deduce for some $\beta<1$ that

$\| z-Bv \| \leq \beta\|z-Bv\| < \|z-Bv\|$,

Which is a contradiction and so Bv=z.

Since Tv = z thus Tv=Bv z. i.e. v is a coincidence point of B and T.

If AK is complete, then by (1), $z\in AK \subset TK$. Similarly if BK is complete, then $z\in BK \subset SK$. Since the pair {A, S} is weakly compatible therefore A and S commute at their coincidence point. Then ASu=SAu or Az = Sz.

Similarly BTv=TBv or Bz = Tz.

Now we prove Az = z.

we have $\| Az-Bx2n_{+1}\|^2 \leq f(\|Sz-Tx2n_{+1}\|^2, \| Sz-Bx2n_{+1}\|\| Tx2n_{+1}-Az\|,$

$\| Sz-Az \|\| Tx2n_{+1}-Az\|)$. By[2]

Taking the limit $n\to\infty$, we have

$\| Az-z \|^2 \leq f(\|Az-z\|^2, \| Az-z\|\| z-Az\|, \| Az-Az \|\| z-Az\|)$

$\leq f(\|Az-z\|^2, \| Az-z\|^2, 0)$

$\leq g(\|Az-z\|)$

$\| Az-z \|^2 \leq \|Az-z\|$

A contradiction, therefore Az=z and Az=Sz=z. similarly we prove Bz=z

Hence Az=Bz =Sz=z. i.e. zis a common fixed point of |A, B, S and T.

For uniqueness let wbe another common fixed point of A, B, S, and T.

By (2), we have $\| Az-Bw \|^2 \leq f(\|Sz-Tw\|^2, \|Sz-Bw\|\|Tw-Az\|,$

$\|Sz-Az \|\| Tw-Az\|)$.

$\| Az-Bw \|^2 \leq f(\|z-w\|^2, \| z-w\|\| w-z\|, \| z-z \|\| w-z\|)$

$\leq f(\|z-w\|) < \|z-w\|$

This gives $\|z-w\|^2 < \|z-w\|$.

A contradiction. Therefore z=w.

This complete the proof of the theorem.

4. Reference

1. Husain S.A, Sehgal V.M. On common fixed point for a family of mapping, Bull. Austr. Math.Soc 1975; 13:261-267.
2. Khan M.S, Imadad M. Fixed point theorem for a class of mapping, Comm. Fac. Soc. Univ. Ankara Sci, 1983, 32.
3. Rashwan R.A. A common fixed point theorem in uniformly convex Banach space, Indian Journal of pure and Applied Mathematics 1998; 3:117-126.
4. Imadad M, khan M.S, Sessa S. On fixed point in uniformly convex Banach spaces, Mathematics Notae, ANO 1984; 31:26-39.
5. Goebel K, Kirk W.A, Shimi T.N. A fixed point theorem in uniformly convex spaces, Bull, Un. Mat. Ital 1973; 4(7):67-75.
6. Sessa S. On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. (Beograd) 1982; 46(32):149-153.
7. Jungck G, Compatible mapping and common fixed points, Internat J Math and math Sci. 1986; 9:771-779.
8. Jungck G, Rhoodes B.E. Fixed point forest vaued functions without continuity, Ind J Pure and Appl Math. 1998; 29(3):227-238.