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Applications of Fourier-Laplace transform to differential equations

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Abstract
Fourier and Laplace Transforms continue to be a very important tool for the engineer, physicist and applied mathematician. They are also now useful to financial, economic and biological modelers as these disciplines become more quantitative. Any problem that has underlying linearity and with solution based on initial values can be expressed as an appropriate differential equation and hence be solved using Fourier and Laplace transform

This paper provides the reader with a solid foundation in the fundamentals of Fourier transforms and Laplace transforms; and gains an understanding of some of the very important and basic applications of these fundamentals to solution of some differential equations.

Keywords: Fourier transform, Laplace transform, Fourier- Laplace transform, generalized function.

1. Introduction

An Integral transform is an operator that maps functions from one space to another. And the practical motivation for an integral transform is to reduce the complexity of the problem. i.e. the mathematical operations will be much easier to handle in the image space. The methods of integral transforms are very efficient to solve and research differential and integral equations of mathematical physics [1].

Among all linear integral transformations, the Laplace transformation stands out as the most versatile in its applications. This may be attributed to the fact that it can convert a system of differential, or integral, or integro-differential equations into algebraic equations [2]. The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

There are many kinds of transforms out there in the world. Laplace and Fourier transforms are probably the main two kinds of transforms that are used. The Fourier and Laplace transforms are great mathematical tools that have served science well for nearly two centuries. Recently however, the Fourier/Laplace transform has become increasingly inadequate for treating one specific class of signals, namely, signals represented by travelling wave equations. The Fourier/Laplace transform is essentially a tool for time domain to frequency domain translation, and vice a versa [3]. These Fourier and Laplace transforms have various uses. Fourier transforms lies at the heart of signal processing, and image processing etc. Laplace transform also have used in solving various partial differential equations. We have combined these two transforms to form Fourier-Laplace Transform in the region ($-\infty$ to ∞) and (0 to ∞). This newly formed Fourier-Laplace integral Transform is used to solve certain partial differential equations.

In this paper we tried to find out the Fourier-Laplace Transform of n^{th} derivatives and by using these results we have solved some partial differential equations such as one dimensional wave equation, Laplace equation in Cartesian form and one dimensional Heat flow equation with boundary conditions.

This paper is summarized as follows:

In Section 2, Preliminary results are given. Fourier-Laplace integral transform of derivatives are given in section 3, in section 4, solved some partial differential equations with examples. Lastly the conclusions are given in section 5.

2. Preliminary Results

The Fourier transform with parameter s of $f(t)$ denoted by $F\left[f(t)\right]=F(s)$ and is given by

$$F\left[f(t)\right]=F(s)=\int_{-\infty}^{\infty}e^{-ist}f(t)dt \text{, for parameter } s>0. \qquad (2.1)$$

The Laplace transform with parameter p of $f(x)$ denoted by $L\left[f(x)\right]=F(p)$ and is given by

$$L\left[f(x)\right]=F(p)=\int_{0}^{\infty}e^{-px}f(x)dx \text{, for parameter } p>0. \qquad (2.2)$$

The Conventional Fourier-Laplace transform is defined as

$$FL\left\{f(t,x)\right\}=F(s,p)=\int_{-\infty}^{\infty}\int_{0}^{\infty}f(t,x)K(t,x)dtdx\,, \tag{2.3}$$

Where,

$K(t,x)=e^{-i(st-ipx)}$.

3. Fourier-Laplace Integral Transform of Derivatives

Theorem: Suppose that $f\left(t,x\right)$ is a continuous for all $y\geq 0$ and $z\geq 0$ satisfying for some value γ,η and m has a derivative $f_1\left(t,x\right)$ which is piecewise continuous on every finite interval in the range of $y\geq 0$ and $z\geq 0$. Then by using the Fourier-Laplace integral transform, the derivative of $f\left(t,x\right)$ exists when $s>\gamma$ and $p>\eta$ and $\left|f\left(t,x\right)\right|\leq me^{\gamma y+\eta z}$ for all $t\geq 0$ and $x\geq 0$ for some constants.

3.1. Fourier-Laplace Integral Transform of first order partial derivative of $f\left(t,x\right)$ w.r.t. x

The Fourier-Laplace transformation is

$$FL\left\{f\left(t,x\right)\right\}=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(st-ipx)}f\left(t,x\right)dtdx\text{, then}$$

$$FL\left\{f_x\left(t,x\right)\right\}=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(st-ipx)}f_x\left(t,x\right)dtdx$$

$$=\int_{-\infty}^{\infty}e^{-ist}dt\int_{0}^{\infty}f_x\left(t,x\right)e^{-px}dx$$

Integrate w.r.t. $'x'$ and t constant

$$=\int_{-\infty}^{\infty}e^{-ist}dt\left[\left(f\left(t,x\right)e^{-px}\right)_0^{\infty}-\int_{0}^{\infty}(-p)e^{-px}f\left(t,x\right)dx\right]$$

$$=\int_{-\infty}^{\infty}e^{-ist}dt\left[-f\left(t,0\right)+p\int_{0}^{\infty}e^{-px}f\left(t,x\right)dx\right]$$

$$=-\int_{-\infty}^{\infty}e^{-ist}f\left(t,0\right)dt+p\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(st-ipx)}f\left(t,x\right)dtdx$$

$$=pFL\left\{f\left(t,x\right)\right\}-\int_{-\infty}^{\infty}e^{-ist}f\left(t,0\right)dt$$

$$\therefore FL\left\{f_x\left(t,x\right)\right\}=pFL\left\{f\left(t,x\right)\right\}-k \tag{3.1.1}$$

where,

$$k=\int_{-\infty}^{\infty}e^{-ist}f\left(t,0\right)dt$$

3.2. Fourier-Laplace integral transform of nth order partial derivative of $f\left(t,x\right)$ w.r.t. x

The Fourier-Laplace integral transform is $FL\left\{f\left(t,x\right)\right\}=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(st-ipx)}f\left(t,x\right)dtdx$, then

$$FL\left\{ f_{xx}\left(t,x\right)\right\} = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i\left(st-ipx\right)} f_{xx}\left(t,x\right)dtdx$$

$$= \int_{-\infty}^{\infty} e^{-ist}dt\int_{0}^{\infty} f_{xx}\left(t,x\right)e^{-px}dx$$

Integrate w.r.t. x and t - constant

$$= \int_{-\infty}^{\infty} e^{-ist}dt\left[\left(f_{x}\left(t,x\right)e^{-px}\right)_{0}^{\infty} - \int_{0}^{\infty}\left(-p\right)e^{-px} f_{x}\left(t,x\right)dx\right]$$

$$= \int_{-\infty}^{\infty} e^{-ist}dt\left[- f_{x}\left(t,0\right)+ p\int_{0}^{\infty} e^{-px} f_{x}\left(t,x\right)dx\right]$$

$$= -\int_{-\infty}^{\infty} e^{-ist} f_{x}\left(t,0\right)dt + p\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i\left(st-ipx\right)} f_{x}\left(t,x\right)dtdx$$

$$= pFL\left\{ f_{x}\left(t,x\right)\right\} - \int_{-\infty}^{\infty} e^{-ist} f_{x}\left(t,0\right)dt$$

$$= pFL\left\{ f_{x}\left(t,x\right)\right\} - 0$$

Where,

$\int_{-\infty}^{\infty} e^{-ist} f_{x}\left(t,0\right)dt = 0$ (By using DUIS- Differentiation under integral sign and it is zero for infinite integral or it is ignore).

$$= p\left\{ pFL\left\{ f\left(t,x\right)\right\} - k\right\}$$

$$FL\left\{ f_{xx}\left(t,x\right)\right\} = p^{2}FL\left\{ f\left(t,x\right)\right\} - pk \qquad (3.2.1)$$

$$FL\left\{ f_{xxx}\left(t,x\right)\right\} = p^{3}FL\left\{ f\left(t,x\right)\right\} - p^{2}k \qquad (3.2.2)$$

$$FL\left\{ f_{x}^{n}\left(t,x\right)\right\} = p^{n}FL\left\{ f\left(t,x\right)\right\} - p^{n-1}k \qquad (3.2.3)$$

This is the generalized result of the Fourier-Laplace Integral transform of n[th] derivative of $f\left(t,x\right)$.

4. Application to limit

4.1. One dimensional wave equation is solved by using Fourier-Laplace Integral Transform

$$\frac{\partial^{2} f}{\partial t^{2}} = c^{2}\frac{\partial^{2} f}{\partial x^{2}}$$

Applying Fourier-Laplace Integral transform to this one dimensional wave equation $f_{tt}\left(t,x\right) = c^{2} f_{xx}\left(t,x\right)$ i.e.

$$FL\left\{ f_{tt}\left(t,x\right)\right\} = c^{2}FL\left\{ f_{xx}\left(t,x\right)\right\}$$

$$FL\left\{ f_{tt}\left(t,x\right)\right\} = c^{2}\left[p^{2}FL\left\{ f\left(t,x\right) - pk\right\}\right]$$

$$D_{t}^{2}FL\left\{ f\left(t,x\right)\right\} = c^{2}\left[p^{2}FL\left\{ f\left(t,x\right)\right\} - pk\right]$$

$$D_{t}^{2}FL\left\{ f\left(t,x\right)\right\} = c^{2}p^{2}FL\left\{ f\left(t,x\right)\right\} - c^{2}pk$$

$$D_{t}^{2}FL\left\{ f\left(t,x\right)\right\} - c^{2}p^{2}FL\left\{ f\left(t,x\right)\right\} = -c^{2}pk$$

$$\left(D_t^2 - c^2 p^2\right) FL\left\{f\left(t,x\right)\right\} = -c^2 pk \tag{4.1.1}$$

this is the ordinary differential equation w.r.t. t in $FL\left\{f\left(t,x\right)\right\}$

$CF = C_1 e^{cpt} + C_2 e^{-cpt}$ And $PI = \frac{k}{p}$

Therefore Complete Solution is

$$FL\left\{f\left(t,x\right)\right\} = C_1 e^{cpt} + C_2 e^{-cpt} + \frac{k}{p}$$

Where,

$$k = \int_{-\infty}^{\infty} e^{-ist} f\left(t,0\right) dt$$

Example

To illustrate the use of the Fourier-Laplace integral transform in solving the certain partial differential equations. We propose to find the solution $f\left(t,x\right)$ of the equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$, satisfying the boundary conditions,

The initial and boundary conditions are

(1) If $t = 0$ then $f\left(0,x\right) = 0$ (2) If $t = a$ then $f\left(a,x\right) = 0$

The solution of one dimensional wave equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$ is $FL\left\{f\left(t,x\right)\right\} = C_1 e^{cpt} + C_2 e^{-cpt} + \frac{k}{p}$

If $t = 0$ then $f\left(0,x\right) = 0 = C_1 + C_2 + \frac{k}{p}$ $\therefore C_1 + C_2 = -\frac{k}{p}$

If $t = a$ then $f\left(a,x\right) = 0 = C_1 e^{cpa} + C_2 e^{-cpa} + \frac{k}{p}$, then $C_1 e^{cpa} + C_2 e^{-cpa} = -\frac{k}{p}$

$$C_1 + C_2 e^{-2cpa} = -\frac{k}{p} e^{-cpa}$$

We get,

$C_1 = -\frac{k}{p}\left(1 - \left(1 + e^{-cpa}\right)^{-1}\right)$ and $C_2 = -\frac{k}{p}\left(1 + e^{-cpa}\right)^{-1}$, then

$$FL\left\{f\left(t,x\right)\right\} = -\frac{k}{p}\left(1 - \left(1 + e^{-cpa}\right)^{-1}\right) e^{cpt} - \frac{k}{p}\left(1 + e^{-cpa}\right)^{-1} e^{-cpt} + \frac{k}{p} \tag{4.1.2}$$

Where,

$$k = \int_{-\infty}^{\infty} e^{-ist} f\left(t,0\right) dt$$

4.2. Laplace equation in Cartesian form is solved by using Fourier-Laplace Integral Transform $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$

The Fourier-Laplace Integral Transformation is

$FL\left\{f\left(t,x\right)\right\} = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i\left(st - ipx\right)} f\left(t,x\right) dt dx$, then

$$FL\left\{f_x\left(t,x\right)\right\} = pFL\left\{f\left(t,x\right)\right\} - k$$

$$FL\left\{f_{xx}\left(t,x\right)\right\} = p^2 FL\left\{f\left(t,x\right)\right\} - pk$$

Where,

$$k = \int_{-\infty}^{\infty} e^{-ist} f(t,0)\,dt$$, then

$$FL\{f_{tt}(t,x)\} + FL\{f_{xx}(t,x)\} = 0$$

$$D_t^2 FL\{f(t,x)\} + p^2 FL\{f(t,x)\} - pk = 0$$

$$D_t^2 FL\{f(t,x)\} + p^2 FL\{f(t,x)\} = pk$$

$$\left(D_t^2 + p^2\right) FL\{f(t,x)\} = pk \qquad (4.2.1)$$

This is the ordinary differential equation of second order in t. Its roots are $m_1 = ip$ and $m_2 = -ip$. Its Complementary function is

$C.F. = C_1 \cos pt + C_2 \sin pt$ And Particular Integral = P.I.= $\frac{k}{p}$, the Complete Solution is

$$FL\{f(t,x)\} = C_1 \cos pt + C_2 \sin pt + \frac{k}{p} \qquad (4.2.2)$$

Where,

$$k = \int_{-\infty}^{\infty} e^{-ist} f(t,0)\,dt$$

Example

To illustrate the use of the Fourier-Laplace integral transform in solving the certain partial differential equations. We propose to find the solution $f(t,x)$ of the equation $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$, satisfying the boundary conditions

(1) If $t = 0$ then $f(0,x) = 0$ (2) If $t = a$ then $f(a,x) = 0$

Answer: - The solution of partial differential equation $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$ is given by

$$FL\{f(t,x)\} = C_1 \cos pt + C_2 \sin pt + \frac{k}{p}$$

(1) If $t = 0$ then $f(0,x) = 0 = C_1 + \frac{k}{p}$ then $C_1 = -\frac{k}{p}$

(2) If $t = a$ then $f(a,x) = 0 = C_1 \cos pa + C_2 \sin pa + \frac{k}{p}$

$$C_2 = -\frac{k}{p}\cos ec pa - C_1 \cot pa = -\frac{k}{p}\cos ec pa + \frac{k}{p}\cot pa$$

Then the required solution is

$$FL\{f(t,x)\} = -\frac{k}{p}\cos(pt) + \left[\frac{k}{p}\cot(pa) - \frac{k}{p}\cos ec pa\right]\sin pt + \frac{k}{p} \qquad (4.2.3)$$

where,

$$k = \int_{-\infty}^{\infty} e^{-ist} f(t,0)\,dt$$

4.3. One dimensional Heat flow Equation is solved by using Fourier-Laplace Integral Transform

The one dimensional Heat flow equation is $\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2}$

The Fourier-Laplace Integral Transformation is

$$FL\left\{ f\left(t,x\right)\right\} = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(st-ipx)} f\left(t,x\right) dt dx$$, then

$$FL\left\{ f_x\left(t,x\right)\right\} = pFL\left\{ f\left(t,x\right)\right\} - k$$

$$FL\left\{ f_{xx}\left(t,x\right)\right\} = p^2 FL\left\{ f\left(t,x\right)\right\} - pk$$, where $k = \int_{-\infty}^{\infty} e^{-ist} f\left(t,0\right) dt$, then

$$FL\left\{ f_t\left(t,x\right)\right\} = c^2 FL\left\{ f_{xx}\left(t,x\right)\right\}$$

$$D_t FL\left\{ f\left(t,x\right)\right\} = c^2\left\{ p^2 FL\left[f\left(t,x\right)\right] - pk\right\}$$

$$D_t FL\left\{ f\left(t,x\right)\right\} = c^2 p^2 FL\left\{ f\left(t,x\right)\right\} - c^2 pk$$

$$\left(D_t - c^2 p^2\right) FL\left\{ f\left(t,x\right)\right\} = -c^2 pk \qquad (4.3.1)$$

This is the linear differential equation in $FL\left\{ f\left(t,x\right)\right\}$ with Integrating factor I.F. is $e^{-c^2 p^2 t}$

The complete solution is

$$FL\left\{ f\left(t,x\right)\right\} e^{-c^2 p^2 t} = \int\left(-c^2 pk\right) e^{-c^2 p^2 t} dt + C$$

$$FL\left\{ f\left(t,x\right)\right\} e^{-c^2 p^2 t} = \frac{k}{p} e^{-c^2 p^2 t} + C$$

$$FL\left\{ f\left(t,x\right)\right\} = \frac{k}{p} + Ce^{c^2 p^2 t} \qquad (4.3.2)$$

$$k = \int_{-\infty}^{\infty} e^{-ist} f\left(t,0\right) dt ,$$

Example

To illustrate the use of the Fourier-Laplace integral transform in solving the certain partial differential equation $\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2}$.

We propose to find the solution satisfying the boundary conditions. Its solution is

$$FL\left\{ f\left(t,x\right)\right\} = \frac{k}{p} + Ce^{c^2 p^2 t}$$

(1) If $t = 0$ then $f\left(0,x\right) = 0$ (2) If $t = a$ then $f\left(a,x\right) = 0$

Answer: - (1) If $t = 0$ then $f\left(0,x\right) = 0 = \frac{k}{p} + C$ $\therefore C = -\frac{k}{p}$ (4.3.3)

$$FL\left\{ f\left(t,x\right)\right\} = \frac{k}{p}\left(1 - e^{c^2 p^2 t}\right) \qquad (4.3.4)$$

OR If $t = a$ then $f\left(a,x\right) = 0 = \frac{k}{p} + Ce^{c^2 p^2 a}$ (4.3.5)

On solving (4.3.1.1) and (4.3.1.3) we get $C = 0$

Then the solution is $FL\left\{ f\left(t,x\right)\right\} = \frac{k}{p}$ (4.3.6)

Where,

$$k = \int_{-\infty}^{\infty} e^{-ist} f\left(t,0\right) dt,$$

5. Conclusion

Since Fourier and Laplace transforms has found numerous applications in various fields. We tried to develop a new type of transform, Fourier-Laplace transform on the same lines. The main aim of this paper is by using Fourier-Laplace integral transform, solution of the Laplace equation in Cartesian form, one dimensional wave equation and Heat flow equations are solved.

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