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On residue product of two fuzzy graphs

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Abstract
In this paper, the residue product of two fuzzy graphs is defined. The effective, connected and complete properties of the residue product are studied. The degree and total degree of a vertex in the residue product of two fuzzy graphs are obtained. It is illustrated that when two fuzzy graphs are regular then their residue product need not be regular. But the conditions under which residue product of two regular fuzzy graphs is regular are given. Also it is proved that the lexicographic max product of G_1 with G_2 is the direct sum of the maximal product and the residue product of the fuzzy graphs if $\sigma_1 \leq \mu_2$.

Keywords: Fuzzy Graph, Effective Fuzzy Graph, Regular Fuzzy Graph, Connectedness and Residue Product.
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Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. Later on, Bhattacharya [1] gave some remarks on fuzzy graphs. Some operations on fuzzy graphs were introduced by Mordeson. J.N. and Peng. C.S. [2]. The conjunction of two fuzzy graphs was defined by Nagoor Gani.A and Radha. K. [3]. We defined the direct sum [4], the strong product [6], the lexicographic products [8], maximal product [9] of two fuzzy graphs and studied the properties of these operations.
In this paper, the residue product of two fuzzy graphs is defined. The effective, connected and complete properties of the residue product are studied. The degree and total degree of a vertex in the residue product of two fuzzy graphs are obtained. The regular and totally regular properties of the residue product are studied. Also it is proved that the lexicographic max product of G_1 with G_2 is the direct sum of the maximal product and the residue product of the fuzzy graphs if $\sigma_1 \leq \mu_2$.

2. Preliminariesa

First let us recall some preliminary definitions that can be found in [1, 11].
A fuzzy graph G is a pair of functions (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. The underlying crisp graph of $G:(\sigma, \mu)$ is denoted by $G^*(V, E)$ where $E \subseteq V\times V$.
Let $G:(\sigma, \mu)$ be a fuzzy graph. The underlying crisp graph of $G:(\sigma, \mu)$ is denoted by $G^*:(V, E)$ where $E \subseteq V\times V$. A fuzzy graph G is an effective fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $(u,v)\in E$ and G is a complete fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u,v\in V$. Therefore G is a complete fuzzy graph if and only if G is an effective fuzzy graph and G* is complete.

The degree of a vertex u of a fuzzy graph G is defined as $d_G(u) = \sum_{u\neq v}\mu(uv) = \sum_{uv\in E}\mu(uv)$ and the total degree is defined as $td_G(u) = d_G(u) + \sigma(u).$

If $d_G(v)=k$ for all $v\in V$, that is, if each vertex of G has the same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph. If $td_G(v)=k$ for all $v\in V$, then G is said to be a totally regular fuzzy graph of degree k or a k-totally regular fuzzy graph. The regular fuzzy graph G is called a full regular fuzzy graph if its underlying crisp graph G* is a regular graph and a complete regular fuzzy graph if its underlying crisp graph G* is a complete graph.
Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ respectively. Define $G:(\sigma, \mu)$ with underlying crisp graph $G^*: (V,E)$ where $V = V_1 \times V_2$, $E = \{(u_1, v_1)(u_2, v_2) / u_1 u_2\in E_1 \text{ or } u_1= u_2 \text{ and } v_1 v_2\in E_2\}$, $\sigma = \sigma_1[\sigma_2]$ and $\mu=\mu_1[\mu_2]$ by, $\sigma(u_1, v_1) = \sigma_1(u_1) \vee \sigma_2(v_1)$, for all $(u_1, v_1) \in V_1 \times V_2$ and

$$\mu\big((u_1,\ v_1)(u_2,\ v_2)\big) = \begin{cases} \mu_1(u_1 u_2) & ,\text{if } u_1u_2 \in E_1 \\ \sigma_1(u_1)\vee \mu_2(v_1v_2) & ,\text{if } u_1 = u_2, v_1v_2 \in E_2. \end{cases}$$

This is called the lexicographic max-product of the fuzzy graph G_1 with G_2 and is denoted by $G_1[G_2]_{max}:(\sigma, \mu)$.
Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ respectively. Define $G:(\sigma, \mu)$, where $\sigma = \sigma_1 * \sigma_2$ and $\mu = \mu_1 * \mu_2$, with underlying crisp graph $G^*: (V, E)$ where $V = V_1 \times V_2$ and $E = \{(u_1, v_1)(u_2, v_2) / u_1= u_2, v_1 v_2\in E_2 \text{ or } v_1= v_2, u_1 u_2\in E_1 \}$, by $\sigma(u_1, v_1) = \sigma_1(u_1) \vee \sigma_2(v_1)$, for all $(u_1, v_1) \in V_1 \times V_2$ and

$$\mu\big((u_1, v_1)(u_2, v_2)\big) = \begin{cases} \sigma_1(u_1) \vee \mu_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \mu_1(u_1u_2) \vee \sigma_2(v_1), \text{if } v_1 = v_2, u_1u_2 \in E_1 \end{cases}$$

This is called the maximal product of the fuzzy graphs G_1 and G_2 and denoted by G_1*G_2.

Notation: The relation $\sigma_1 \leq \sigma_2$ means that $\sigma_1(u) \leq \sigma_2(v)$ for every $u \in V_1$ and for every $v \in V_2$ where σ_i is a fuzzy subset of V_i, i =1,2.

3. Residue Product

3.1. Definition

Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ respectively. Define $G:(\sigma,\mu)$, with underlying crisp graph G^*: (V,E) where $V = V_1 \times V_2$ and $E = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E_1, v_1 \neq v_2\}$, by $\sigma(u_1, v_1) = \sigma_1(u_1) \vee \sigma_2(v_1)$, for all $(u_1,v_1) \in V$ and $\mu\big((u_1,v_1)(u_2,v_2)\big) = \mu_1(u_1 u_2), \text{for all} \big(u_1,v_1\big)\big(u_2,v_2\big) \in E.$

If $u_1 u_2 \in E_1$ and $v_1 \neq v_2$ then,

$\mu((u_1, v_1)(u_2, v_2)) = \mu_1(u_1 u_2) \leq \sigma_1(u_1) \wedge \sigma_1(u_2) \leq [\sigma_1(u_1) \vee \sigma_2(v_1)] \wedge [\sigma_1(u_2) \vee \sigma_2(v_2)] = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$.

Hence $\mu((u_1, v_1)(u_2, v_2)) \leq \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$. Therefore, $G:(\sigma, \mu)$ is a fuzzy graph. This is called the residue product of the fuzzy graphs G_1 and G_2 and denoted by $G_1 \bullet G_2$.

3.2. Example

The following Figure-1 illustrates the residue product $G_1 \bullet G_2$ of the two fuzzy graphs G_1 and G_2.

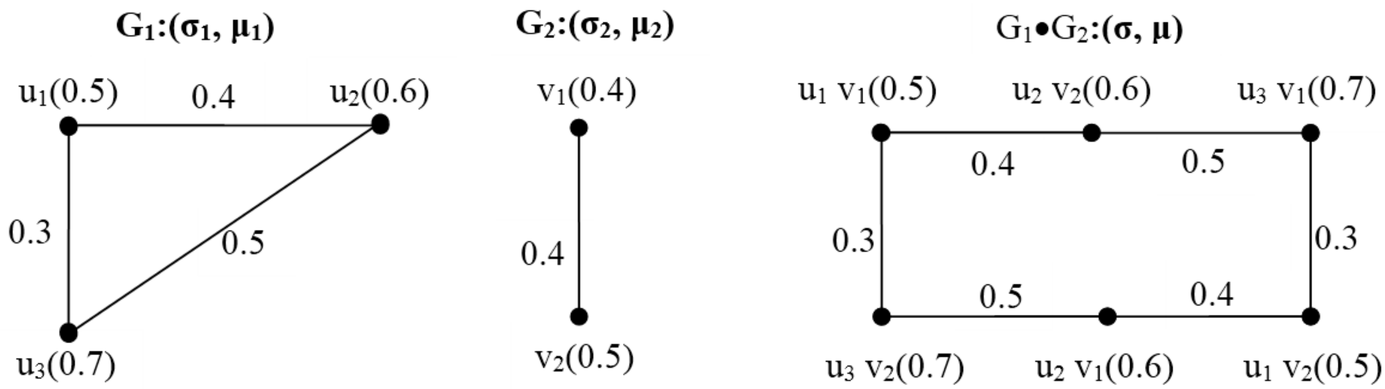

Fig 1

3.3. Theorem

The residue product of an effective fuzzy graph $G_1:(\sigma_1,\mu_1)$ with any fuzzy graph $G_2:(\sigma_2,\mu_2)$ is an effective fuzzy graph if $\sigma_1 \geq \sigma_2$.

Proof

Let $G_1:(\sigma_1,\mu_1)$ be an effective fuzzy graph and $G_2:(\sigma_2,\mu_2)$ be any fuzzy graphs with $\sigma_1 \geq \sigma_2$.

Then $\mu_1(u_1u_2) = \sigma_1(u_1) \wedge \sigma_1(u_2)$ for any $u_1u_2 \in E_1$. Then proceeding as in the definition,

If $u_1 u_2 \in E_1$ and $v_1 \neq v_2$ then,

$\mu((u_1, v_1)(u_2, v_2)) = \mu_1(u_1 u_2) = \sigma_1(u_1) \wedge \sigma_1(u_2)$

$= [\sigma_1(u_1) \vee \sigma_2(v_1)] \wedge [\sigma_1(u_2) \vee \sigma_2(v_1)] = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$.

Thus $\mu((u_1, v_1)(u_2, v_2)) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$ for all edges in the residue product.

Hence $G_1 \bullet G_2$ is an effective fuzzy graph.

3.4. Example

Consider the two fuzzy graphs G_1 and G_2 given in the following Figure-2. G_1 is an effective fuzzy graph with $\sigma_1 \geq \sigma_2$ and their residue product $G_1 \bullet G_2$ is also an effective fuzzy graph.

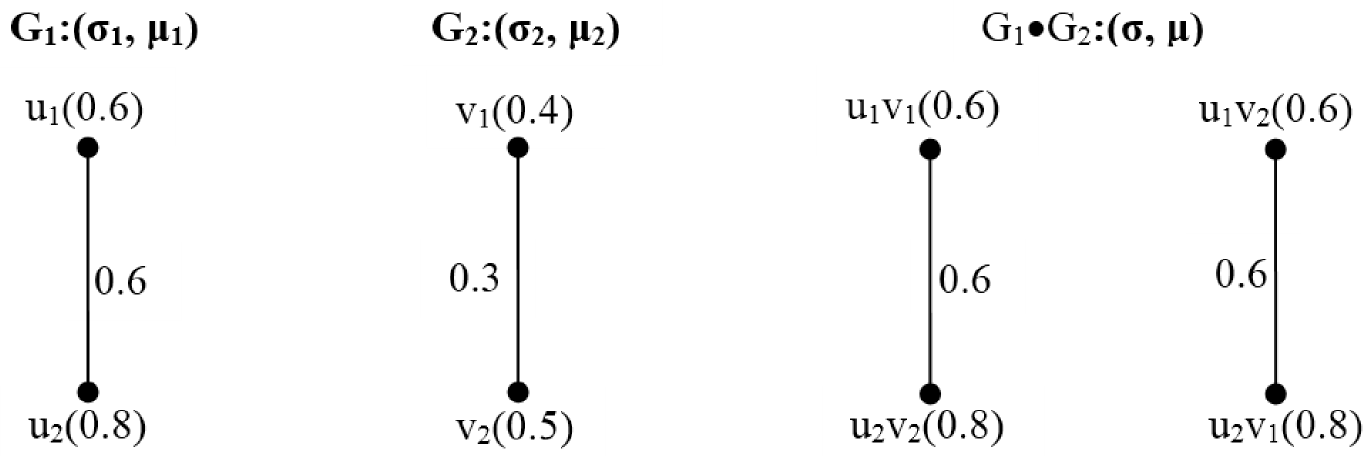

Fig-2

3.5. Remark

The residue product of two connected fuzzy graphs need not be a connected fuzzy graph.

Consider the two connected fuzzy graphs G_1 and G_2 given in Figure-2 of Example3.4. Their residue product $G_1 \bullet G_2$ is not a connected fuzzy graph.

3.6. Remark

The residue product of two complete fuzzy graphs is not a complete fuzzy graph because we include only the case $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$ in the definition of the residue product.

Since every complete fuzzy graph is effective, from Theorem 2.3, we have the residue product of a complete fuzzy graph $G_1:(\sigma_1,\mu_1)$ with any fuzzy graph $G_2:(\sigma_2,\mu_2)$ is an effective fuzzy graph if $\sigma_1 \geq \sigma_2$.

Consider the following Figure-3 where G_1 is a complete fuzzy graph and G_2 is a fuzzy graph with $\sigma_1 \geq \sigma_2$. Their residue product $G_1 \bullet G_2$ is an effective fuzzy graph.

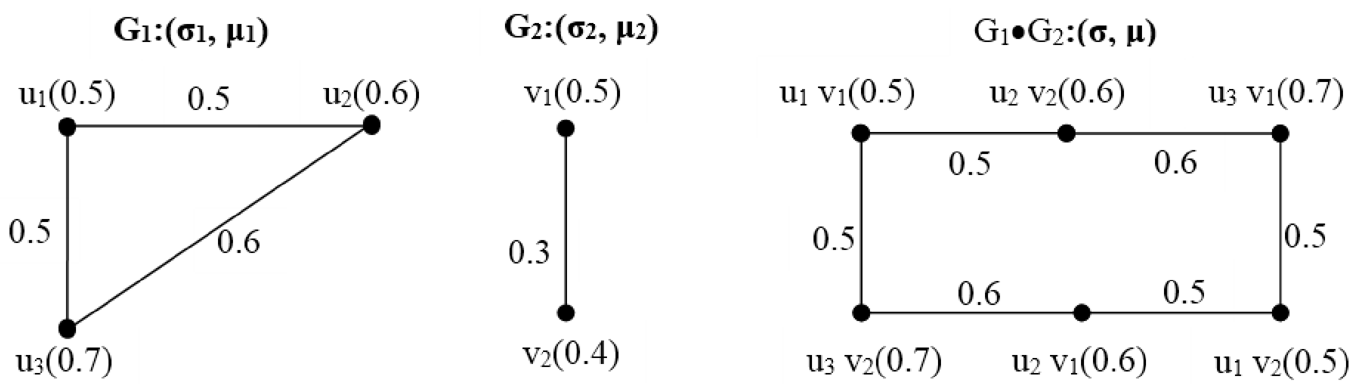

Fig 3

4. Regular Property of Residue Product

The degree of any vertex in the residue product $G_1 \bullet G_2$ of the fuzzy graph $G_1:(\sigma_1,\mu_1)$ with $G_2:(\sigma_2,\mu_2)$ is given by,

$$d_{G_1 \bullet G_2}(u_i, v_j) = \sum_{u_i u_k \in E_1, v_j \neq v_\ell} \mu_1(u_i u_k) = d_{G_1}(u_i)$$

4.1. Example

Consider the residue product of the two fuzzy graphs $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ given in Figure-1 of Example 2.2. The degree of any vertex in the residue product is given by,

$$d_{G_1 \bullet G_2}(u_1, v_j) = 0.4 + 0.3 = 0.7 = d_{G_1}(u_1) \text{ for } j = 1, 2.$$

$$d_{G_1 \bullet G_2}(u_2, v_j) = 0.5 + 0.4 = 0.9 = d_{G_1}(u_2) \text{ for } j = 1, 2.$$

$$d_{G_1 \bullet G_2}(u_3, v_j) = 0.5 + 0.3 = 0.8 = d_{G_1}(u_3) \text{ for } j = 1, 2.$$

Now consider the residue product of the two fuzzy graphs $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ given in the following Figure-4.

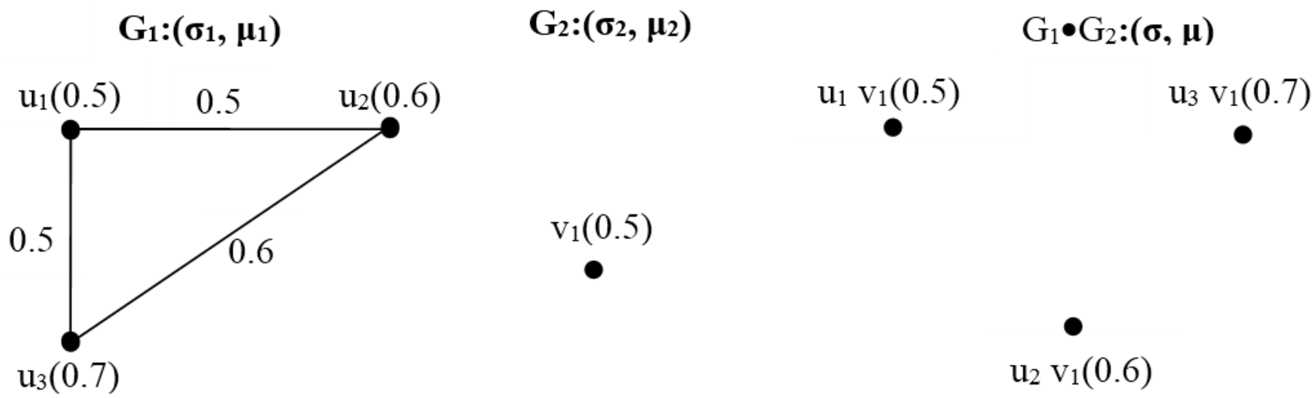

Fig 4

The degree of any vertex in the residue product $G_1 \bullet G_2$ of the fuzzy graph $G_1:(\sigma_1,\mu_1)$ with $G_2:(\sigma_2,\mu_2)$ is given by, $d_{G_1 \bullet G_2}(u_i, v_1) = 0, \text{ for } i = 1, 2, 3.$ Hence the residue product $G_1 \bullet G_2$ of any fuzzy graph $G_1:(\sigma_1,\mu_1)$ with a fuzzy graph $G_2:(\sigma_2,\mu_2)$ such that $|V_2|=1$, is always a fuzzy graph with no edge.

4.2. Theorem

If $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ are two fuzzy graphs such that $|V_2|>1$ then their residue product is regular if and only if G_1 is regular.

Proof

Let $G_1:(\sigma_1,\mu_1)$ be a k-regular fuzzy graph and $G_2:(\sigma_2,\mu_2)$ be any fuzzy graph with $|V_2|>1$.

If $|V_2|>1$, then, $d_{G_1 \bullet G_2}(u_i, v_1) = d_{G_1}(u_i) = k$.

This is a constant for all vertices in $V_1 \times V_2$. Hence $G_1 \bullet G_2$ is a regular fuzzy graph.

Conversely assume that $G_1 \bullet G_2$ is a regular fuzzy graph. Then, for any two vertices (u_1,v_1) and (u_2,v_2) in $V_1 \times V_2$,

$$d_{G_1 \bullet G_2}(u_1, v_1) = d_{G_1 \bullet G_2}(u_2, v_2)$$

$$\Rightarrow \qquad d_{G_1}(u_1) = d_{G_2}(u_2)$$

This is true for all vertices in V_1. Hence G_1 is a regular fuzzy graph.

4.3. Example

Consider the following figure where $G_1:(\sigma_1, \mu_1)$ is a regular and $G_2:(\sigma_2, \mu_2)$ is a fuzzy graph with $|V_2|>1$. Their residue product $G_1 \bullet G_2$ is also a regular fuzzy graph.

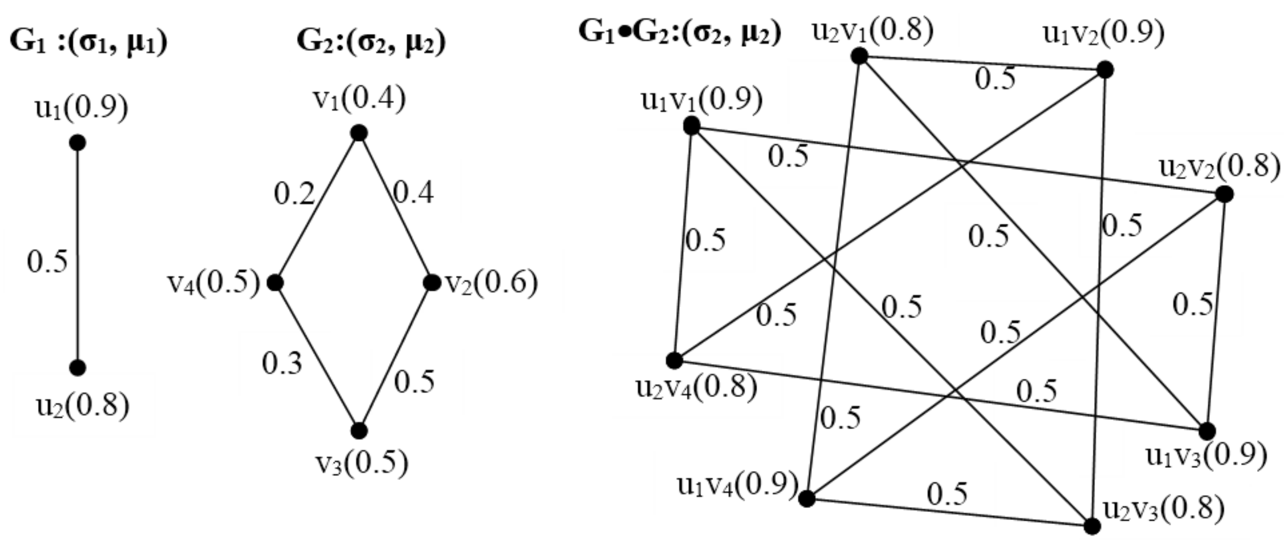

Fig 5

4.4. Remark

The total degree of any vertex in the residue product $G_1 \bullet G_2$ of the fuzzy graph $G_1:(\sigma_1,\mu_1)$ with $G_2:(\sigma_2,\mu_2)$ is given by, $td_{G_1 \bullet G_2}(u_i, v_j) = \sum_{u_i u_k \in E_1, v_j \neq v_\ell} \mu_1(u_i u_k) + \sigma(u_i, v_j)$

4.5. Theorem

If $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ are two fuzzy graphs such that $\sigma_1 \geq \sigma_2$ then the total degree of any vertex in the residue product $G_1 \bullet G_2$ is given by,

$$td_{G_1 \bullet G_2}(u_i, v_j) = \begin{cases} td_{G_1}(u_i), & \text{if } |V_2| > 1 \\ \sigma_1(u_i), & \text{if } |V_2| = 1 \end{cases}$$

Proof

If $\sigma_1 \geq \sigma_2$ and $|V_2|>1$, then,

$$td_{G_1 \bullet G_2}(u_i, v_j) = \sum_{u_i u_k \in E_1, v_j \neq v_\ell} \mu_1(u_i u_k) + \sigma(u_i, v_j)$$

$$= \begin{cases} d_{G_1}(u_i) + \sigma_1(u_i), & \text{if } |V_2| > 1 \\ \sigma_1(u_i), & \text{if } |V_2| = 1 \end{cases}$$

$$= \begin{cases} td_{G_1}(u_i), & \text{if } |V_2| > 1 \\ \sigma_1(u_i), & \text{if } |V_2| = 1 \end{cases}$$

4.6. Theorem

If $G_1:(\sigma_1,\mu_1)$ is a totally regular fuzzy graph and $G_2:(\sigma_2,\mu_2)$ is a fuzzy graph such that $\sigma_1 \geq \sigma_2$ and $|V_2|>1$ then their residue product is also a totally regular fuzzy graph with the same degree.

Proof

Let $G_1:(\sigma_1,\mu_1)$ be a k-totally regular fuzzy graph and $G_2:(\sigma_2,\mu_2)$ be a fuzzy graph such that $\sigma_1 \geq \sigma_2$ and $|V_2|>1$. Then,

$td_{G_1}(u_i) = k$, for all u_i in V_1 and $\sigma(u_1,v_1) = \sigma_1(u_1)$ for all (u_1,v_1) in $V_1 \times V_2$.

Now,

$$td_{G_1 \bullet G_2}(u_1, v_1) = d_{G_1 \bullet G_2}(u_1, v_1) + \sigma(u_1, v_1)$$

$$= d_{G_1}(u_1) + \sigma_1(u_1)$$

$$= td_{G_1}(u_1) = k.$$

This is a constant for all vertices in $V_1 \times V_2$. Hence $G_1 \bullet G_2$ is a k-totally regular fuzzy graph.

4.7. Example

Consider the residue product of the totally regular fuzzy graph $G_1:(\sigma_1,\mu_1)$ with a fuzzy graph $G_2:(\sigma_2,\mu_2)$ such that $|V_2|>1$ given in the following Figure-5. Their residue product is a totally regular fuzzy graph.

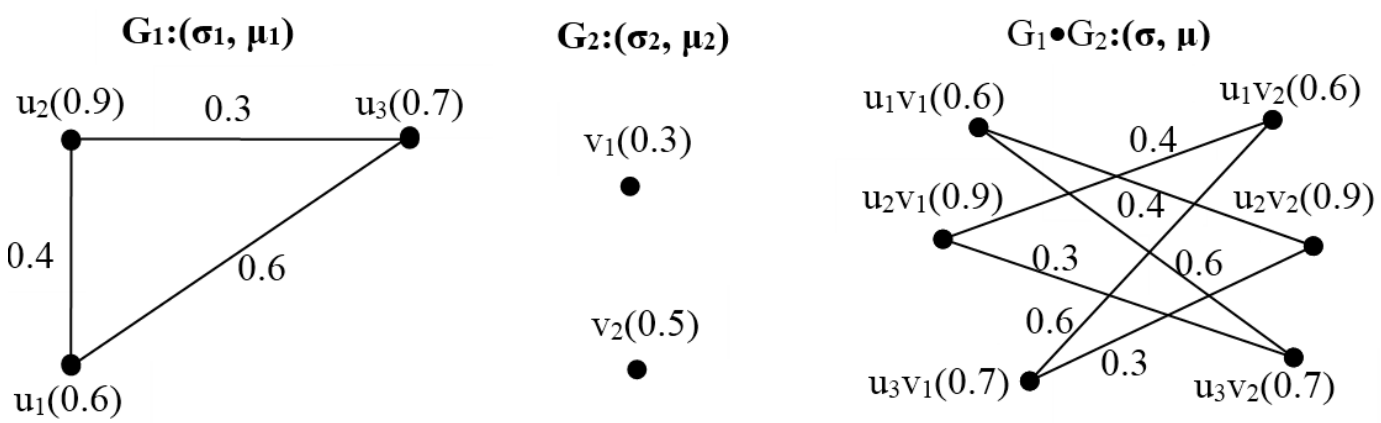

Fig 6

5. Relationship between Lexicographic Max Product and Direct Sum

5.1. Theorem

The lexicographic max product of $G_1:(\sigma_1,\mu_1)$ with $G_2:(\sigma_2,\mu_2)$ is the direct sum of the maximal product and the residue product of the fuzzy graphs if $\sigma_2 \le \mu_1$.

That is, $G_1[G_2]_{max} = (G_1*G_2) \oplus (G_1 \bullet G_2)$ if $\sigma_2 \le \mu_1$.

Proof

Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs such that $\sigma_2 \le \mu_1$. Then $\sigma_2 \vee \mu_1 = \mu_1$. We have the maximal product, $G_1*G_2:(\sigma',\mu')$ where $\sigma' = \sigma_1 \vee \sigma_2$,

$$\mu'((u_1, v_1)(u_2, v_2)) = \begin{cases} \sigma_1(u_1) \vee \mu_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \mu_1(u_1u_2) \vee \sigma_2(v_1), \text{if } v_1 = v_2, u_1u_2 \in E_1 \end{cases}$$

and the residue product $G_1 \bullet G_2:(\tau,\nu)$ of the fuzzy graph $G_1:(\sigma_1,\mu_1)$ with $G_2:(\sigma_2,\mu_2)$ where $\tau = \sigma_1 \vee \sigma_2$,

$$\nu((u_1, v_1)(u_2, v_2)) = \mu_1(u_1 u_2), \text{for all } u_1u_2 \in E_1 \text{ and } v_1 \ne v_2.$$

Now consider $G:(\sigma,\mu) = (G_1*G_2) \oplus (G_1 \bullet G_2)$. Then $\sigma = \sigma_1 \vee \sigma_2$ and

$$\mu((u_1, v_1)(u_2, v_2)) = \begin{cases} \sigma_1(u_1) \vee \mu_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \mu_1(u_1u_2) \vee \sigma_2(v_1), \text{if } v_1 = v_2, u_1u_2 \in E_1 \\ \mu_1(u_1u_2) \quad\quad , \text{if } v_1 \ne v_2, u_1u_2 \in E_1 \end{cases}$$

$$= \begin{cases} \sigma_1(u_1) \vee \mu_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \mu_1(u_1u_2) \quad\quad , \text{if } v_1 = v_2, u_1u_2 \in E_1 \\ \mu_1(u_1u_2) \quad\quad , \text{if } v_1 \ne v_2, u_1u_2 \in E_1 \end{cases}$$

$$= \begin{cases} \mu_1(u_1u_2) \quad\quad , \text{if } u_1u_2 \in E_1 \\ \sigma_1(u_1) \vee \mu_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \end{cases}$$

This $G(\sigma, \mu)$ is nothing but the lexicographic max product $G_1[G_2]_{max}$ of $G_1:(\sigma_1,\mu_1)$ with $G_2:(\sigma_2,\mu_2)$. Thus we have $G_1[G_2]_{max} = (G_1*G_2) \oplus (G_1 \bullet G_2)$ if $\sigma_2 \le \mu_1$.

5.2. Remark

Similarly we can prove that the lexicographic max product of $G_2:(\sigma_2,\mu_2)$ with $G_1:(\sigma_1,\mu_1)$ is the direct sum of the maximal product and the residue product of the fuzzy graphs if $\sigma_1 \le \mu_2$.

That is, $G_2[G_1]_{max} = (G_2*G_1) \oplus (G_2 \bullet G_2)$ if $\sigma_1 \le \mu_2$.

5.3. Example

The following Figure-7 illustrates the theorem 5.1.

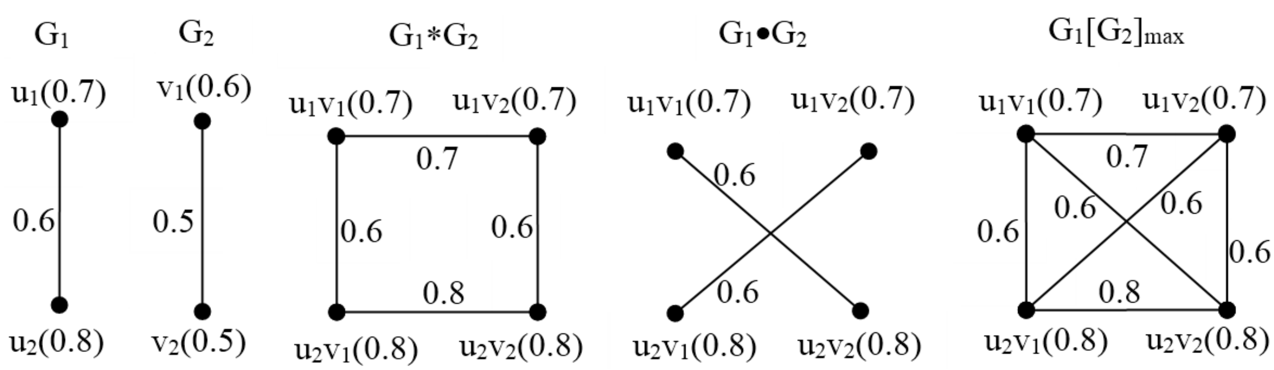

Fig 7

6. Conclusion

In this paper, the residue product of two fuzzy graphs is defined. The effective, connected and complete properties of the residue product are studied. The degree and total degree of a vertex in the residue product of two fuzzy graphs are obtained. It is illustrated that when two fuzzy graphs are regular then their residue product need not be regular. But the conditions under which residue product of two regular fuzzy graphs is regular are given. Also it is proved that the lexicographic max product of G_1 with G_2 is the direct sum of the maximal product and the residue product of the fuzzy graphs if $\sigma_1 \le \mu_2$. In addition to the existing ones, this operation will be helpful to study large fuzzy graph as a combination of small fuzzy graphs and to derive its properties from those of the small ones.

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