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Analysis of the existence and continuity of global attractors in Riemannian wave equations with localized damping

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Abstract

This study addresses the existence and continuity of global attractors for wave equations on Riemannian manifolds, considering the effect of localized damping. Wave equations with localized dissipation represent a relevant model in physical problems, such as wave propagation in media with partial dampers. The following questions are posed: Do exponential global attractors exist for such systems? Is it possible to ensure the continuity of these attractors in response to external perturbations in the system? Using functional analysis techniques and semigroup theory, the existence of global attractors is demonstrated, and their continuity is analyzed.

Keywords: Riemannian manifolds, wave equations, localized damping, global attractors, existence of attractors, mathematical modeling of physical systems

Introduction

The analysis of global attractors in wave equations has become essential for modeling complex physical and structural phenomena. An iconic example illustrating this need is the collapse of the Tacoma Narrows Bridge in 1940, caused by wind-induced resonance. This event highlighted the vulnerability of curved structures to external dynamic interactions, underscoring the importance of detailed studies of vibratory systems with localized damping. See [7] and [8].

Fig 1: Tacoma Narrows Bridge Collapse

Demonstrates structural resonance caused by external forces and damping inadequacy. Dynamically modeled as: $u_{tt} + a(x)u_t - \Delta u = f(x)$, where $a(x)$ is the damping coefficient and f(x) external forcing.

Global Attractors in Wave Equations

Global attractors represent asymptotic states satisfying: $A = \{\phi \in H^1(M) / T(t)\phi$ is precompact and attracts bounded sets$\}$, where $T(t)$ is the semigroup of the system evolution [6].

In the realm of wave equations defined on Riemannian manifolds, such as bridges or other structures with curved geometries, global attractors play a fundamental role in describing the final states toward which dynamic solutions tend. These attractors not only allow for the evaluation of solution stability but also indicate how they react to perturbations or changes in the system's initial and external conditions.

Recent research has addressed the existence of exponential global attractors and their continuity using mathematical tools such as Hausdorff metrics and the effects of localized damping (Babin & Vishik, 1992; Chueshov, 2002) [1, 5]. These advancements have not only deepened the theory of dynamical systems but also contributed to improving the design and stability of engineering structures.

This study aims to demonstrate the existence and continuity of global attractors in wave equations within curved domains, considering the geometric properties of Riemannian manifolds and the impact of localized damping. This approach not only expands theoretical foundations but also has direct applications in analyzing and designing modern infrastructures such as bridges, buildings, and vehicles.

Materials and Methods

Problem Statement

The general research problem is: How can the existence and continuity of global attractors in Riemannian wave equations with localized damping be ensured?

Specific problems include:

1. Do exponential global attractors exist?
2. Is it possible to ensure their continuity in response to external variations?

General Objective: To demonstrate the existence and continuity of global attractors for the proposed equations.

Specific Objectives:

1. To prove the existence of uniformly bounded global attractors.
2. To analyze their continuity using Hausdorff metrics.

The research is theoretically justified as it extends the theory of global attractors to non-Euclidean metrics and optimizes the use of control regions through ε-controllability. Its practical impact lies in applications in structural dynamics and engineering (bridges, buildings) and the simulation of waves in curved domains for stability analysis.

Delimitations:

- **Theoretical:** Focuses on mathematical analysis based on semilinear wave equations and specific tools such as semigroup theory and global attractors.
- **Spatial:** Uses compact manifolds with smooth boundaries (M, g) and localized damping in regions ω ⊂ M.
- **Temporal:** Studies long-term dynamics (exponential attraction) with numerical resolution in controlled domains.

Theoretical Framework

Global attractors in dynamical systems describe the asymptotic behavior of solutions. Mathematically, these trajectories are often governed by systems of differential equations such as:

where are the system variables, and the functions determine the dynamics.

Geometric Control Condition (GCC): The GCC is a fundamental criterion in the study of exact controllability and stabilization of partial differential equation systems, such as wave equations. It establishes a relationship between the support of damping or control and the geometric properties of the domain where the problem is posed.

Fundamental Theorems

Methodological Design: This research is basic, aimed at generating new knowledge in the analysis of differential equations in non-Euclidean geometries. The theoretical approach involves rigorous mathematical analysis consisting of the following stages:

1. **Model Formulation:**

Statement of the equation (P_β) and Definition of global attractors.

2. **Existence and Boundedness:** Application of semigroup theory to prove stability.

3. **Continuity:** Analysis in the Hausdorff metric under variations of β.

4. **Optimization:** Construction of ε-controllable regions in curved manifolds.

- **Problem Statement**:

Global attractors require solutions $u(t)$ to converge within compact subsets as $t \to \infty$.

- **Hausdorff Metric for Continuity**

Measures the distance between attractors under parameter perturbations:

$$d_H(\mathcal{A}_1,\mathcal{A}_2)=\max\{\sup_{x\in\mathcal{A}_1}\inf_{y\in\mathcal{A}_2} d(x,y),\ \sup_{y\in\mathcal{A}_2}\inf_{x\in\mathcal{A}_1} d(x,y)\}$$

The methodology employs deductive-inductive methods based on theoretical reviews and rigorous demonstrations to validate hypotheses.

Results

1. Existence of global attractors

The research demonstrated the existence of global attractors for the Riemannian wave equations with locally optimal damping in measure. These attractors are compact stabilization regions for the dynamic system. Using theoretical frameworks such as the compactness and invariance of sets, it was proven that the system exhibits a generalized exponential attractor with finite fractal dimensions.

2. Semicontinuity of Attractors

The study verified the upper semicontinuity of the global attractors concerning the parameter β. Mathematically, this means that as β approaches a fixed value β_0, the Hausdorff distance between attractors A_β and A_{β_0} tends to zero:

3. Observability and control

The regions of damping were shown to meet optimal conditions under the concept of ϵ\epsilonϵ-controllability, ensuring efficient stabilization of vibrations within compact subsets.

4. Mathematical modeling

The problem was modeled as a semilinear wave equation over a compact Riemannian manifold with boundary:

$$u_{tt} - \Delta_g + a(x)u_t + f(u) = \beta h(x)$$

, where:

$a(x)$ represents the damping coefficient localized in specific regions,

$f(u)$ describes nonlinear forces,

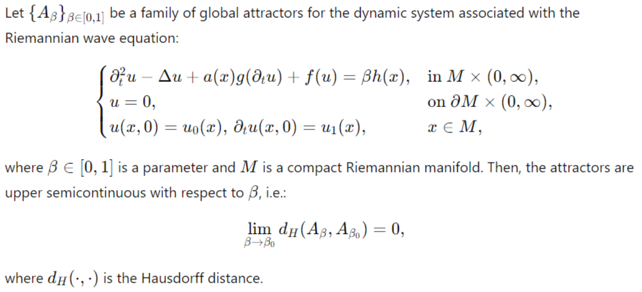
$h(x)$ models external forces.

The solutions of this equation were proven to exhibit long-term stability.

5. Theoretical Contributions

The study expanded the understanding of wave dynamics on curved domains with non-conservative forces. This approach enables the strategic placement of damping regions, optimizing the measure for stability.

Theorem Upper Semicontinuity



Let $\{A_\beta\}_{\beta\in[0,1]}$ be a family of global attractors for the dynamic system associated with the Riemannian wave equation:

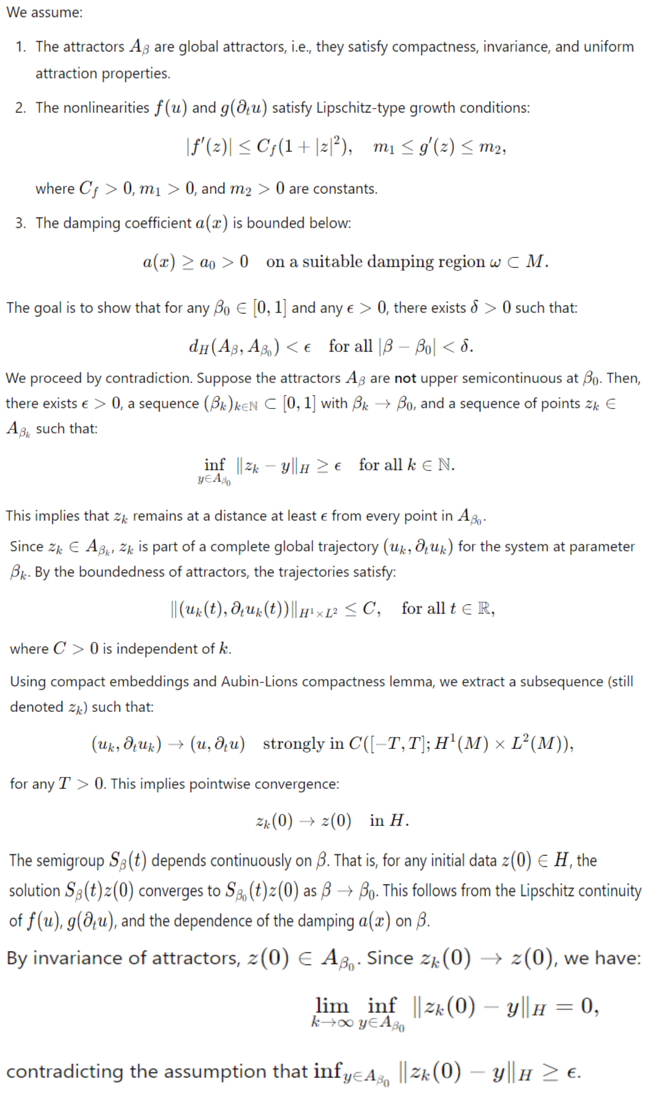
$$\begin{cases} \partial_t^2 u - \Delta u + a(x)g(\partial_t u) + f(u) = \beta h(x), & \text{in } M \times (0,\infty), \\ u = 0, & \text{on } \partial M \times (0,\infty), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), & x \in M, \end{cases}$$

where $\beta \in [0,1]$ is a parameter and M is a compact Riemannian manifold. Then, the attractors are upper semicontinuous with respect to β, i.e.:

$$\lim_{\beta\to\beta_0} d_H(A_\beta, A_{\beta_0}) = 0,$$

where $d_H(\cdot,\cdot)$ is the Hausdorff distance.

Proof:



We assume:

1. The attractors A_β are global attractors, i.e., they satisfy compactness, invariance, and uniform attraction properties.
2. The nonlinearities $f(u)$ and $g(\partial_t u)$ satisfy Lipschitz-type growth conditions:

$$|f'(z)| \leq C_f(1+|z|^2), \quad m_1 \leq g'(z) \leq m_2,$$

where $C_f > 0$, $m_1 > 0$, and $m_2 > 0$ are constants.

3. The damping coefficient $a(x)$ is bounded below:

$$a(x) \geq a_0 > 0 \quad \text{on a suitable damping region } \omega \subset M.$$

The goal is to show that for any $\beta_0 \in [0,1]$ and any $\epsilon > 0$, there exists $\delta > 0$ such that:

$$d_H(A_\beta, A_{\beta_0}) < \epsilon \quad \text{for all } |\beta - \beta_0| < \delta.$$

We proceed by contradiction. Suppose the attractors A_β are **not** upper semicontinuous at β_0. Then, there exists $\epsilon > 0$, a sequence $(\beta_k)_{k\in\mathbb{N}} \subset [0,1]$ with $\beta_k \to \beta_0$, and a sequence of points $z_k \in A_{\beta_k}$ such that:

$$\inf_{y\in A_{\beta_0}} \|z_k - y\|_H \geq \epsilon \quad \text{for all } k \in \mathbb{N}.$$

This implies that z_k remains at a distance at least ϵ from every point in A_{β_0}.

Since $z_k \in A_{\beta_k}$, z_k is part of a complete global trajectory $(u_k, \partial_t u_k)$ for the system at parameter β_k. By the boundedness of attractors, the trajectories satisfy:

$$\|(u_k(t), \partial_t u_k(t))\|_{H^1\times L^2} \leq C, \quad \text{for all } t \in \mathbb{R},$$

where $C > 0$ is independent of k.

Using compact embeddings and Aubin-Lions compactness lemma, we extract a subsequence (still denoted z_k) such that:

$$(u_k, \partial_t u_k) \to (u, \partial_t u) \quad \text{strongly in } C([-T,T]; H^1(M) \times L^2(M)),$$

for any $T > 0$. This implies pointwise convergence:

$$z_k(0) \to z(0) \quad \text{in } H.$$

The semigroup $S_\beta(t)$ depends continuously on β. That is, for any initial data $z(0) \in H$, the solution $S_\beta(t)z(0)$ converges to $S_{\beta_0}(t)z(0)$ as $\beta \to \beta_0$. This follows from the Lipschitz continuity of $f(u)$, $g(\partial_t u)$, and the dependence of the damping $a(x)$ on β.

By invariance of attractors, $z(0) \in A_{\beta_0}$. Since $z_k(0) \to z(0)$, we have:

$$\lim_{k\to\infty} \inf_{y\in A_{\beta_0}} \|z_k(0) - y\|_H = 0,$$

contradicting the assumption that $\inf_{y\in A_{\beta_0}} \|z_k(0) - y\|_H \geq \epsilon$.

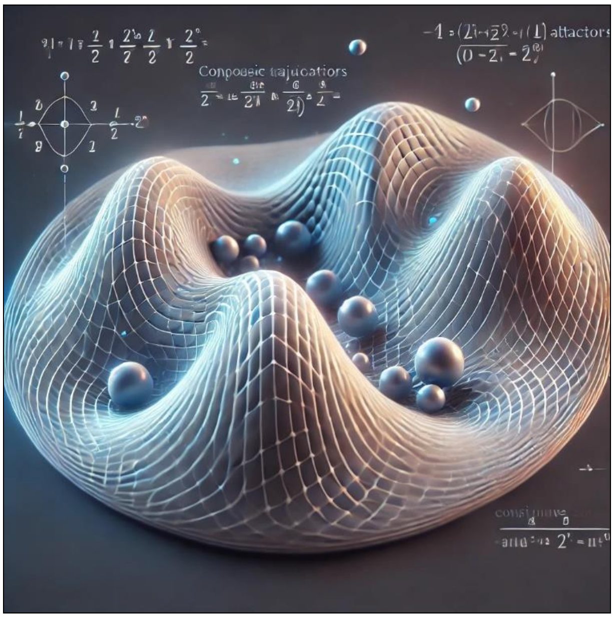


Fig 2: Riemannian manifold with continuous attractors

Conclusions

1. Existence of global attractors

The study confirmed the existence of global attractors for semilinear Riemannian wave equations with optimal localized damping. These attractors represent compact stabilization regions within the dynamic system.

2. Semicontinuity of attractors

The upper semicontinuity of the attractors with respect to the parameter β was proven. This means that small variations in the parameter lead to small changes in the attractors, measured via the Hausdorff distance.

3. Optimal control through damping:

The placement of damping regions in the spatial domain ensures stabilization in measure. The concept of ϵ\epsilonϵ-controllability provides a mathematical framework for optimizing these regions for stability, even in critical Sobolev settings.

4. Innovative context:

The research extends classical wave equation studies to curved Riemannian domains, offering a novel perspective for analyzing dynamic systems in structural engineering and applied mathematics. This includes applications to prevent structural collapse under non-conservative forces.

5. Mathematical rigor and novelty:

The proof techniques and theorems developed in this study introduce new methodologies for demonstrating the stability and continuity of attractors in infinite-dimensional systems, contributing significantly to the field of mathematical physics and dynamic systems.

6. Practical implications:

The findings have potential applications in engineering, particularly in designing damping systems to stabilize vibrations in structures like bridges or curved architectures.

References

1. Babin AV, Vishik MI. Attractors of Evolution Equations. North-Holland, 1992.
2. Bardos C, Lebeau G, Rauch J. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM Journal on Control and Optimization,1992:30(5):1024-1065.
3. Billah KY, Scanlan RH. "Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks." American Journal of Physics,1991:59(2):118–124.
4. Burq N. Contrôle de l'équation des ondes dans des ouverts peu réguliers. Asymptotic Analysis,1998:14(2):157-191.
5. Chueshov ID. Introduction to the Theory of Infinite-Dimensional Dissipative Systems. Springer, 2002.
6. Lions JL. Exact Controllability, Stabilization and Perturbations of Distributed Systems. SIAM Review,1988:30(1):1-68.
7. Ma TF, Seminario-Huertas PN. "Attractors for semilinear wave equations with localized damping and external forces", Communications on Pure ans Applied Analysis,2020:19(9):5111-5130.
8. Mendoza, *et al*. "Global Riemannian wave attractors and the optimal measure of localized damping" Proceedings of the LACCEI international Multiconference for Engineering, Education and Technology, 2023.