International Journal of Multidisciplinary Research and Development
www.allsubjectjournal.com
Online ISSN: 2349-4182, Print ISSN: 2349-5979
Received: 22-06-2022, Accepted: 06-07-2022, Published: 23-07-2022
Volume 9, Issue 7, 2022, Page No. 85-92

The curvature of fiber bundles

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Abstract

Curvature in mathematics is the rate of direction and is the amount by which the geometric surface itself derives from being a straight line. A fiber bundle is a localized as a multiplication space, but which as have a different topological structure as a whole. In differential geometry the curvature form describes curvature of a connection on a principal bundle.

Our goal in this paper is to identify the concept of curvature in fiber bundles, we followed the analytical historical mathematical method and we found that curvature is a scalar quantity that can be determined and most material things are curvature or midwife curvature.

Keywords: curvature, fiber bundles, vector bundles, radius, center of curvature

Introduction

Curvature in mathematics, the rate of direction if a curve with respect to distance along the curve. At every point on a circle, the curvature is the reciprocal of the radius; for other curves (and straight lines, which can be regarded as circles of infinite radius), the curvature is the reciprocal of the radius of the circle that most closely conforms to the curve at the given point. If the curve is a section of a surface (that is, the curve formed by the intersection of a plane with the surface), then the curvature of the surface at any given point can be determined by suitable sectioning planes. The most useful planes are two that both contain the normal (the line perpendicular to the tangent plane) to the surface at the point. One of these planes produces the section with the greatest curvature among all such sections; the other produces that with the least. These two planes define the two so-called principal directions in the surface at the point; these directions lie at right angles to one another. The curvatures in the principal directions are called the principal curvatures of the surface. The mean curvature of the surface at the point is either the sum of the principal curvatures or half that sum (usage varies among authorities). The total (or Gaussian) curvature is the product of the principal curvatures. One of the most basic results in Euclidean geometry is that the sum of the angles of a triangle is 180 degrees, or in other word, the sum of two right angles. Recall the proof. Given a triangle with vertices p, Q and R, by Playfair's axiom there is a unique line through R parallel to the line spanned by P and Q. by results on alternating angles, we see that the angles α, β and γ must sum to that of two right angles.

Note that we needed to use Play fair's axiom. Thus this result will not necessarily be true in non-Euclidean geometries. This seems reasonable if we look at the picture of a triangle in the hyperbolic upper half-plane and of a triangle on the sphere of double elliptic geometry.

What happens is that in hyperbolic geometry the sums of the angles of a triangle are less than280 degrees while, for elliptic geometries, the sum of the angles a triangle will be greater than 180 degrees. It can be shown that the smaller that the area of the triangle is, the closer the sum of the triangle's angle will be to 180 degrees. This in turn is linked to the Gaussian curvature. It is the case (though it is not obvious) that methods of geometry (i.e., metrics) can be chosen so that the different types of geometry will have different Gaussian curvatures. More precisely, the Gaussian curvature of the Euclidean plane will be zero, of the hyperbolic plane will be -1 and of the elliptic planes will be 1. Thus differential geometry and curvature are linked to the axiomatic of different geometries.

The following geometric figures (1), (2) show the relationship of differential geometry to curvature.

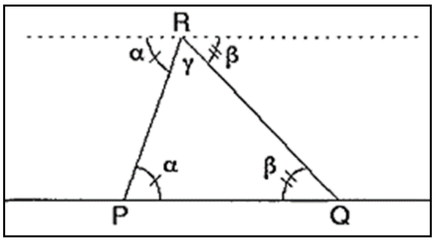

Fig 1

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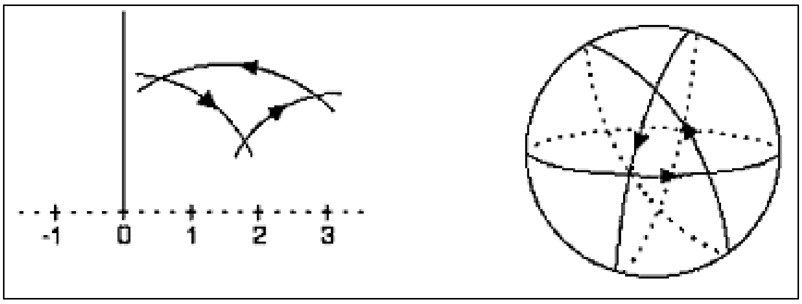

Fig 2

What happens is that in hyperbolic geometry the sums of the angles of a triangle are less than180 degrees while, for elliptic geometries, the sum of the angles a triangle will be greater than 180 degrees. It can be shown that the smaller that the area of the triangle is, the closer the sum of the triangle's angle will be to 180 degrees. This in turn is linked to the Gaussian curvature. It is the case (though it is not obvious) that methods of geometry (i.e., metrics) can be chosen so that the different types of geometry will have different Gaussian curvatures. More precisely, the Gaussian curvature of the Euclidean plane will be zero, of the hyperbolic plane will be -1 and of the elliptic planes will be 1. Thus differential geometry and curvature are linked to the axiomatic of different geometries. According to mathematics curvature is any of the number loosely related concepts in different areas of geometry. Naturally, it is the amount by which geometric surfaces derivate themselves from being flat plane and also from a curve being straight like a line.

However, it is defined differently from different context.

Furthermore, the curvature is a scalar quantity that one can also define as a curvature vector that takes into explanation the direction of the curve and its magnitude. Moreover, the curvature of the complex object is distinct by complex objects from linear algebra.

Center of Curvature

We considered a point (x_0, y_0) on an arc of a curve whose equation is $y = f(x)$ and for which the curvature is positive, the arc lying in the first quadrant. But it may be shown that the formulae obtained for the coordinates, (x_c, y_c), of the center of curvature apply in any situation; provided that the curvature is associated with its appropriate sing.

Note that although the formulae apply in any situation, it is a good idea to sketch the curve in order estimate, roughly, where the center of curvature is going to be. This is especially important where there is uncertainty about the precise value of the angle θ.

Example 2.1: [11] determine the center of curvature at the point (0.5, -1) of the curve whose equation is

Conclusion

According to the results of this current study, we can be concluded that nonnutritive sucking stimulation can decrease in the incubation period for low birth infants by increasing their abilities to suck.

Conflict of interest

The authors have no conflicts of interest to declare.

$$= 2ry^2$$

Solution

From the center example on calculating radius of curvature,

$\frac{1}{y}$ and $\frac{d^2y}{dx^2} = -\frac{1}{y^3}, \frac{dy}{dx}$

Giving $\frac{dy}{dx}$ = -1, $\frac{d^2y}{dx^2}$ = 1 and ρ= $2\sqrt{2}$ at the point (0.5, -1).

Radius of Curvature

The radius of curvature 'R' in differential geometry is the reciprocal of the curvature. Besides the radius of the circular arc is the best approximate the curve at that point. Also, for surfaces, the radius of the curvature is the radius of the circle that best in a normal section or combination.

Formula of the Radius of Curvature: Normally the formula of curvature is as:

$$R = 1/K \qquad (1)$$

Here K is the curvature. Also, at a given point R is the radius of the osculating circle (an imaginary circle that we draw to know the radius of curvature).
Besides, we can sometimes use symbol ρ in lace of R for the denotation of a radius of curvature.

Application of Radius of Curvature: In differential geometry, it is used in Cesaro equation which tells that a plain curve is an equation that relates the curvature (K) at a point of the curve to the arc length (s) from the start of the curve to a given point. Also, it is an equation relating to the radius of curvature (R) to the arc length. Also it can help to find the radius of curvature of the earth along a course at an azimuth. Besides, the radius of curvature also uses three parts equation for bending of beams. Moreover, it has a specific meaning and a sign convention in optical design. Also, spherical lenses have a center of curvature.

Difference between Radius and Radius of Curvature: Radius refers to the distance between the center of a circle or any other point on the circumference of the circle and surface of the sphere. While on the other hand, the radius of curvature is the radius of the circle that touches the curve at a given point. Also, it has the same tangent and curvature at that point. Moreover, the radius is or a real figure or shape whereas the radius of curvature is an imaginary circle.

Types of Curvatures: In general there are two types of curvature namely extrinsic curvature and intrinsic curvature. In this topic, we will discuss these two types.

1. **Extrinsic Curvature:** It a curvature that is a sub manifold of manifold that depends on its particular inserting. Besides, its examples include the torsion of curves in three-space and the curvature. Also, it includes the mean curvature of surfaces in three-space.

2. **Intrinsic Curvature:** It is also a curvature such as Gaussian curvature that is detectable to two dimensional the" populations" of a surface and not just outside observers. Also, they cannot study three dimensional.

The Curvature Form

Definition 8.1: [8]
Let π: P→M be a principal G-bundle with a connection H, and let ω be the connection form of H. Then the g-valued 2-form *Ω*= DꞶ is called the curvature form of H. The connection H is said to be flat if *Ω*= 0.

Proposition 8.2: [8]
Let π: P→M be a principal G-bundle over a 1-dimensional smooth manifold M. Then every connection H in P is flat.

Proof
If *Ω* is the curvature form of H, we must show that *Ω* (u) (w_1, w_2)=0 for every u ∈ P and w_1, w_2∈ TuP. Since M is 1-dimensional, we know that dim (H_u)= 1.

If {w} is a basis for H_u and h (w_i)= c_iw for i= 1, 2, we have that

$$(u)\ (w_1, w_2) = c_1, c_2\ dw\ (u)\ (w_1, w_2) = 0 \qquad (2)$$

Since dw (u) is skew symmetric.

Proposition 8.3: [8]
Let π: P→M be a principal G-bundle with a connection H, and let ω and *Ω* be the connection form and the curvature form of H. Then we have that

$$\Omega(X, Y) = dw\ (h\circ X, h\circ Y) = -Ꞷ\ ([h\circ X, h\circ Y]) \qquad (3)$$

For any pair of vector fields X, Y on an open subset U of P.

Corollary 8.4: [8]
Let H be a connection in principal G-bundle π: P→M and let *Ω* be the curvature form of H. Then H is an integrable distribution on P if and only if *Ω* = 0.

Proof
If H is integrable, then [h○X, h○Y] ∈H which implies than *Ω*(X, Y) = 0 for every pair of vector fields X, Y on P.

Conversely, assume that $\Omega = 0$, for any pair of vector fields X, Y ∈ H defined on some open subset U of P, it follows from proposition (8.2) that [X, Y] = [h○X, h○Y] ∈H, which shows that H is integrable distribution on P. It well-know that the exterior differentiation in the algebra of differential forms on a manifold satisfies dd =0. Let us compute in our general case value of ddα for α ∈$\mathfrak{A}^0$(C, M) = M. Then one has

$$dd\alpha\,(X, Y) = D_X(d\alpha)(Y) - D_Y(d\alpha)(X) - (d\alpha)([X, Y]) \qquad (4)$$

$$= D_X D_{Y\alpha} - D_Y D_{X\alpha} - D_{[X,Y]}\alpha$$

Let K (X, Y) = $D_X D_Y - D_Y D_X - D_{[X,Y]}$. Then it is obvious that K (X, Y) is a k- endomorphism of M. But, it actually follows in trivial verification that it is an A- endomorphism. On the other hand, we have

1. K (X, Y) = −K (Y, X)
2. K (X+X', Y) =K (X, Y) + K(X',Y), and
3. K (aX, Y) α = aK (X, Y) α for every α ∈M.

1.-and 2.- are trivial and we shall verify only 3.-.

$$K\,(aX, Y) = D_{aX} D_Y - D_Y D_{aX} - D_{[aX,Y]}$$
$$= aD_X D_Y - D_Y(aD_X) - D_{a[X,Y]} - (Ya)\,X$$
$$= aD_X D_Y - (Ya)D_X - aD_Y D_X - aD_{[X,Y]} + (Ya)D_X$$
$$= a\,K\,(X, Y).$$

We have therefore proved that *K* is an alternate form of degree 2 over *C* with values in the module Hom_A(M, M).

Definition 8.5: [4]

The element *K* of $\mathfrak{A}^2$(C,Hom_A(M, M)) as defined above i.e.,

$$K\,(X, Y) = D_X D_Y - D_Y D_{X-[X,Y]} \qquad (5)$$

is called the curvature form of the derivation law D.

Example 8.6: [4]

1. Take the simplest case when M = A, with the canonical derivation law. Then

$$K\,(X, Y)\,u = XYu - Yxu - [X, Y]\,u = 0 \qquad (6)$$

i.e., the curvature form is identically zero.

2. However, there are examples in which the curvature form is nonzero.

Let A =k[x, y] with x, y transcendental over k. It is easy to see that *C* is the free module over A with P$(=\frac{\partial}{\partial x})$, Q $(=\frac{\partial}{\partial y})$ as base.

We take M to be A itself but with a derivation law different form the canonical one. By Th1, Ch.1.2, if we choose 1 ∈ M and take ω ∈ Hom_A(C, A), then there exists a derivation law D such that $D_X 1$= ω (X) We shall define ω by requiring that ω (P) = y and ω (Q) = 1. Then it follows that $D_P(1)$ = y, $D_Q(1)$ =1.

$$K\,(P, Q)\,(1) = D_P\, D_Q(1) - D_Q\, D_P(1) - D_{[P,Q]}(1)$$
$$= y - (Qy).1 - y\,(D_Q(1))$$
$$= -1.$$

Lemma 8.7: [4]

α − $\theta_Y \theta_X \alpha = \theta_{[X,Y]}\alpha$ + K(X, Y)(α) for every α ∈$\mathfrak{A}^P$(C, M). $\theta_X\theta_Y$

In fact, when α is of degree 0, the formula is just definition of the curvature form. In the general case, this follows on straight forward verification.

Lemma 8.8: [4]

dα − $\theta_Y \theta_X$ α =$\theta_{[X,Y]}\alpha$+ K(X,Y)(α) for every α ∈$\mathfrak{A}^P$(C, M). θ_X

It will be noted that K $\in\mathfrak{A}^2$(C, Hom_A(M, M) and hence 1_XK has values in $Hom_A(M, M)$. Taking M_1= Hom_A(M, M), M_2= M, M_3= M in our standard notation, one has a bilinear product $M_1 \times M_2 \rightarrow M_3$defined by (h, u). The symbol $\wedge$ used in the enunciation of the lemma is with reference to this bilinear product.
Proof: As usual, we prove this by induction on p, the degree of α. When α is of degree 0, the formula reduces to

dα (u)) – dα ([X, u]) – (dD_Xα)(u) = (1_XK ∧ α) (u) for u ∈ C. D_X(

i.e., $D_X D_u \alpha - D_{[X,u]}\,\alpha - D_u D_X \alpha = (1_X K \wedge (\alpha)\,(u)$

which is but the definition of K. Assuming the truth of the lemma for forms of degree < p, we have
$d\alpha - 1_Y d\theta_X \alpha - 1_Y((1_X K \wedge \alpha)$
$= \theta_X 1_Y d\alpha - 1_{[X,Y]} d\alpha + d1_Y \theta_X \alpha - \theta_Y \theta_X \alpha - (1_Y 1_X K) \wedge \alpha + (1_X K) \wedge (1_Y \alpha)$
$= -\theta_X d1_Y \alpha + \theta_X \theta_Y \alpha - 1_{[X,Y]} d\alpha + d\theta_X 1_Y \alpha - d1_{[X,Y]} \alpha - \theta_Y \theta_X \alpha - (1_Y 1_X K) \wedge \alpha + (1_X K) \wedge (1_Y \alpha)$
$= -\theta_X d1_Y \alpha + d\theta_X 1_Y \alpha + \theta_X \theta_Y \alpha - \theta_Y \theta_X \alpha - \theta_{[X,Y]} \alpha - (1_Y 1_X K) \wedge \alpha + (1_X K) \wedge (1_Y \alpha)$
$= -\theta_X d1_Y \alpha + d\theta_X 1_Y \alpha + (1_X K) \wedge (1_Y \alpha)$
= 0 by induction assumption.

Fiber Bundles

A natural mathematical framework for gauge filed theories is that of the fiber bundle formalism. It effects a clear separation of the kinematic, supplied by the structure of the base manifold, which usually represents a physical space or space time and the dynamics, supplied by the specification of a Lagrangian. In the case of Yang-Mills theories, for instance the group of symmetries of the Lagrangian, the internal symmetry, is made local by constructing a fiber bundle with the fiber being the symmetry group. This, then, enables sections, a connection and curvature to be defined on the bundles which represent physical fields on the base manifold, or physical space time. Although general relativity cannot be formulated as a Yang- Mills theory, it can also be formulated using fiber bundles, making the theory of fibre bundles a uniform framework to treat all the fundamental fields in modern physics.
A bundle is defined to be a structure (E,π, M) consisting of smooth manifold E, a smooth manifold M and on onto smooth map π: E→M. the manifold E is called the projection map. It is common to refer to the bundle as E, when it is clear what the base space and projection map are. For each point p $\in$M, the invers image to p under π

E_p= {q$\in$ E: $\boldsymbol{\pi}$ (q) =p} (7)

Is called the fiber over p. thus, we have that

E = $U_{p\in M} E_p$. (8)

If for each p$\in$ME_p is homeomorphic to a common space F, then F is known as the fiber of the bundle and the bundle is called a fibre bundle. An important example of a fibre bundle is the tangent bundle TM formed from the set of all tangent spaces T_pM of

M: TM= $U_{p\in M} T_p$M, (9)

Where the projection map $\boldsymbol{\pi}$: TM→M is the map from each tangent vector $v_p \in T_p$M to the point p $\in M$ and the fibre of the bundle is R^n

Definition 9.1: [3] With the above notations the bundle (X_F,p_F, B), denoted ξ [F], is called the fiber bundle over B with fiber F and associated principal bundle ξ. The group G is called the structure group of the fiber bundle ξ [F].
Roughly speaking, a principal G-bundle ξ = (X, p, B) consists of a copy of G for each point b ∈ B all "glued together" by the topology of X. the associated fiber bundle ξ [F] consists of a copy of F for each point of B all "glued together" in a manner prescribed by the topology of the total space X, the action of G on X, and the action of G on F. this gluing is done using the quotient space X × F mod G.

Curvature of a Connection

Definition 10.1: [7] The notion of curvature and torsion are well known from elementary Geometry. In our current general framework we define the curvature of a connection in an abstract way.

Definition 10.2: [5] For any connection ∇ on a smooth vector bundle E → M, the object $(d^{\nabla})^2 \in \Omega^2(End_{\mathbb{K}}(E))$ is called the curvature of ∇, and it is usually denoted by F (∇).

Example 10.3: [5] Consider the trivial bundle $\mathbb{K}^r_M$. The sections of this bundle are smooth $\mathbb{K}^r$–valued functions on M. the exterior derivative d defines the trivial connection on $\mathbb{K}^r_M$, and any other connection differs from d by a M_r($\mathbb{K}$)valued 1-form on M. if A is such a form, then the curvature of the connection d + A is the 2-form F (A) defined by

$$F(A)u = (d + A)^2 u = (dA + A \wedge A)u, \forall u \in C^\infty(M, \mathbb{K}^r).$$

The $\wedge$-operation above is defined for any vector bundle E as the bilinear map

$$(\mathrm{End}(E)) \times \Omega^k(\mathrm{End}(E)) \to \Omega^{j+k}(\mathrm{End}(E)), \Omega^j$$

Uniquely determined by

$$(\omega^r \otimes A) \wedge (\eta^s \otimes B) = \omega^j \wedge \eta^k \otimes AB, A, B \in C^\infty(\mathrm{End}(E)).$$

Curvature of Connection in a Fiber Bundle

Definition 11.1: (Horizontal forms in a fibred manifold) in a fiber manifold p $\in C^1(\mathbb{E}, M)$, a form K $\in \Lambda^K(\mathbb{E}, T\mathbb{E})$ is horizontal if it vanishes when any of its arguments is a vertical tangent vector to T$\mathbb{E}$. This concept is independent of the choice of a connection.

Curvature on a Vector Bundle

If E→ M is a vector bundle and ∇ is a linear connection it is natural to ask whether covariant derivative operators ∇_X and ∇_Y in different direction commute. of course this is not even true in general for the Lie derivatives L_X and L_Y on C^∞(M), which one can view as the trivial connection on a trivial line bundle. Their lack of commutatively can however be measured via the identity

$$L_X L_Y - L_Y L_X = L_{[X,Y]}, \qquad (10)$$

And one might wonder whether it is true generally that $\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$ as operators on Γ (E). The answer turns out to be no in general, but the failure of this identity can be measured precisely in terms of curvature.

Definition 12.1: [6] Given a linear connection ∇ on a vector bundle $E \to M$, the curvature tensor is the unique multilinear bundle map

$R: TM \oplus TM \oplus E \to E: (X, Y, \upsilon) \mapsto R(X, Y)\upsilon$
Such that for all X, Y ∈ Vec (M) and $\upsilon \in \Gamma$ (E),
$$R(X, Y)\upsilon = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\upsilon \qquad (11)$$

Theorem12.2: [6]
For any vector bundle E →M with connection ∇, the curvature tensor R satisfies

$$R(X, Y)\upsilon = -K([X_h, Y_h](\upsilon)) \qquad (12)$$

for any vector fields X, Y ∈ Vec (M) and $\upsilon \in$ E.
Note that K on the right-hand side of this formula is not quite the same projection as in the previous section: as is standard for linear connection, K is now the connection map *K:* TE →E obtained from the vertical projection via the identification $V_\upsilon E = E_p$ for $\upsilon \in E_p$. In light of this, it is natural to give a slightly new (but equivalent) definition of the curvature 2-form when the connection is linear. We define an antisymmetric bilinear bundle map Ω_k: TM ⊕ TM → End (E) by the formula

$$\Omega_k(X, Y)\upsilon = -K([X_h, Y_h](\upsilon)), \qquad (13)$$

For any p ∈ M, X, Y $\in T_pM$ and $\upsilon \in E_p$, where on the right hand side we choose arbitrary extensions of X and Y to vector fields near p. it's straightforward to check that this expression is C^∞- linear in both X and Y; what's less obvious is that it is also linear with respect to υ. This is true because the connection map K: TE → E satisfies K o $Tm_\lambda = m_\lambda$o K, where m_λ: E → E is the map $\upsilon \mapsto \lambda\upsilon$ for any scalar $\lambda \in \mathbb{F}$. Indeed, since m_λ is a diffeomorphism on E whenever $\lambda \neq 0$, we have $(m_\lambda)_* [\xi, \eta] \equiv [(m_\lambda)_*\xi, (m_\lambda)_*\eta]$ for any $\xi, \eta \in$ Vec (E), thus

$$\begin{aligned}(X, Y)(\lambda\upsilon) &= -K([X_h, Y_h](\lambda\upsilon)) = -K([X_h, Y_h] \circ m_\lambda(\upsilon)) \,\Omega_k \\ &= -K(Tm_\lambda \circ [X_h, Y_h](\upsilon)) = -m_\lambda \circ K([X_h, Y_h](\upsilon)) \\ &= \lambda.\, \Omega_k(X, Y)\upsilon.\end{aligned}$$

With our new definition of the curvature 2−form, theorem (5.9.1) can be restated succinctly as

$$R(X, Y)\, v = \Omega_k(X, Y)\, v \qquad (14)$$

It follows that $\nabla_X \nabla_Y v - \nabla_Y \nabla_X v - \nabla_{[X,Y]} v \equiv 0$ for all vector fields X, Y and sections v if and only if the connection is flat. As an important special case, if α (s, t) ∈ M is a smooth map parameterized by two real variables and v (s, t) $\in E_{\alpha(s,t)}$ defines a smooth section of E along α, we have

$$v \equiv \nabla_t \nabla_s v \nabla_s \nabla_t$$

If and only if ∇ is flat; more generally

$$v - \nabla_t \nabla_s v = R\,(\partial_s \alpha, \partial_t \alpha)\, v . \nabla_s \nabla_t$$

We shall prove theorem (5.9.1) by relating the bracket to an exterior derivative using a generalization of the standard formula

$$D\alpha\,(X, Y) = L_X(\alpha(Y)) - L_X(\alpha(X)) - \alpha([X, Y]) \qquad (15)$$

For 1- forms α $\in \Omega^1$(M). In particular, the definitions of Ω_k, K and R can all be expressed in terms of bundle-valued differential forms. For any vector bundle π: E →M, define

$$(M, E)\Omega^k$$

To be the vector space of smooth real multilinear bundle maps

$$\omega : \underbrace{TM \oplus \cdots \oplus TM}_{k} \to E$$

which are ant symmetric in the k variables. By this definition, Ω^k(M) is simply Ω^k(M, M ×ℝ), i.e. the space of k-forms taking values in the trivial real line bundle.

Proof of theorem (5.9.1):
We will show that both R (X, Y) v and Ω_k(X, Y) v can be expressed in terms of a covariant exterior derivative of the connection map K: TE →E. in this context, we regard K as a bundle-valued 1-form K $\in \Omega^1(E, \pi^* E)$, and use the connection ∇ on π: E →M to induce a natural connection on the pullback bundle $\pi^* E$→ E. this is the unique connection such that for any smooth path γ (t) ∈ E and section v (t) $\in \pi^* E_{\gamma(t)} = E_{\pi o \gamma(t)}$ along γ, $\nabla_t v$ matches the covariant derivative of v as a section of E along the path $\pi o \gamma(t)$∈M.
We claim first that for any p ∈ M, $v \in E_p$ and X, Y $\in T_p$M,

$$K\,(Hor_v\,(X), Hor_v\,(Y)) = \Omega_k(X, Y)\, v . d_\nabla$$

Indeed, extend X and Y to vector fields on M and use the corresponding horizontal lifts X_h, $Y_h \in$ Vec (E) as extension of Hor_v (X), $and\, Hor_v$ (Y) $\in T_v$E respectively. Then using (5.5) and the fact that K vanishes on horizontal vectors

$$\begin{aligned} K\,(X_h(v), Y_h(v)) &= \nabla_{X_h(v)}(K\,(Y_h)) - \nabla_{Y_h(v)}(K\,(X_h)) - K\,([X_h, Y_h](v)\, d_\nabla \\ &= \Omega_K(X, Y)\, v. \end{aligned}$$

We now show that R (X, Y) v can also be expressed in this way. Choose a smooth map α(s, t) ∈ M for (s, t) $\in \mathbb{R}^2$ near (0, 0) such that ∂_sα(0, 0) =X and ∂_tα(0, 0) =Y, and extend $v \in E_p$ to a section v (s, t) $\in E_{\alpha(s,t)}$ along α such that $\nabla_s v(0, 0) = \nabla_t v(0, 0) = 0$. Then expressing covariant derivatives via the connection map (e.g. $\nabla_s v = K\,(\partial_s v)$) and applying (5.5) once more, we find

$$\begin{aligned} R\,(X, Y)\, v &= \nabla_s \nabla_t v\,(0, 0) - \nabla_t \nabla_s v\,(0, 0) \\ &= \nabla_s(K\,(\partial_s v\,(s, t))) - \nabla_t(K\,(\partial_t v\,(s, t))) \Big|_{(s,t)=(0,0)} \\ &= d_\nabla K\,(\partial_s v, \partial_T v) = d_\nabla K\,(Hor_v(X), Hor_v(Y)) \end{aligned}$$

Where in the last step we used the assumption that *v (s, t)* has vanishing covariant derivatives at *(0, 0)*.

Results

We obtained these results: curvature is the amount by which a curve deviates from straight line. It is defined in a way which relates to the tangent angle and the arc length of the curve, A fiber bundle is a topological space with different structure, and each bundle consists of a continuous immersive application π: E→B.

Conclusion

We dealt with the definitions of curvature its center, radius, types and its relationship to connection and then we talked about the fiber bundles and the curvature in the vector bundles and we found a set of results.

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