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Operational calculus for two dimensional fractional fourier-Mellin transform

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Abstract
An integral transform is useful if it allows one to turn a complicated problem into a simpler one. The Fourier-Mellin transformation has applications as registration of images, watermarks, invariant pattern recognition, preprocessing of images, registration of medical images, the comparison of plant leaves, reconstruction of the grayscale images, detecting watermark in images regardless of the scaling and rotation, detection of human face etc.
In present work we discuss about shifting operator, scaling operator and shifting-scaling operator. Also present some theorems on differential operator.

Keywords: Two dimensional fractional Fourier transform, two dimensional fractional Mellin transform, Two-Dimensional Fractional Fourier-Mellin Transform, Testing function space, Generalized function.

1. Introduction

Integral transform have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics and engineering science. Fourier transform diagonalizes all linear time-invariant operators, which are building blocks of signal processing and many other branches The Fractional Fourier transform (FRFT) is a mathematical tool which maps a signal from one domain to other in image frequency Plane. The application areas for FRFT are explored in the field of electromagnetic wave propagation, radar communication, signal processing and image processing. Various patents were available in literature for the application of FRFT in radar communication, image processing, cryptography and optical focusing.
The Mellin transform has important applications in the solution of boundary value problems in wedge shaped regions. It is also one of the most important methods for the study of classes of functions de_ned on the positive real line. The theory of Mellin transform requires the introduction of new concepts of derivative and integral, called M-derivative and M-integral [3].
Due to shift invariant property of Fourier transform and scaled invariant property of Mellin transform, the Fourier-Mellin transform (FMT) is a very powerful tool in image restoration, pattern recognition. G.J. Pratt used the Fourier-Mellin transformation for the comparison of plant leaves [9]. J. R. Martinez-de Dios and A. Ollero discussed about a robust real-time image stabilization system based on the Fourier-Mellin transform. This system is capable of performing image capture-stabilization-display at a rate of standard video on a general Pentium III at 800 MHz without any specialized hardware and the use of any particular software platforms [5]. A combined Fourier–Mellin transform yields a representation of a signal that is independent of delay and scale change. Such a representation should be useful for speech analysis, where delay and scale differences degrade the performance of correlation operations or other similarity measures [2]. S. Derrode, F. Ghorbel used the approximation of Fourier-Mellin transformation for the reconstruction of the grayscale images [1]. A further development in the use of the Fourier-Mellin transform is its application into the radar classification of ships by Zwicke *et al.* Fourier-Mellin transform is used to identify plant leaves at various life stages based on the leaves shape or contour. Fourier-Mellin transform is also used in estimation of optical flow [4]. Fourier-Mellin transform used the for detection of watermark in images regardless of the scaling and rotation [6]. Using this transform human face also detected [7]. G.S. Page used this method for comparing of distorted objects [8].
In our previous work we define some terminology [4, 10, 11] is as follows

1.1 Definition of two-dimensional fractional Fourier-Mellin transform

The two-dimensional fractional Fourier-Mellin transform with parameters α and θ of $f(x,y,t,q)$ denoted by 2DFRFMT$\{f(x,y,t,q)\}$ performs a linear operation, given by the integral transform. 2DFRFMT
$\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x,y,t,q)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)dxdydtdq$---- (1)

where $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1} q^{\frac{2\pi i\chi}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$

$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{1\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha} C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ $0 < \alpha < \frac{\pi}{2}, 0 < \theta < \frac{\pi}{2}$. -------------------- (2)

1.2 The Test Function

An infinitely differentiable complex valued smooth function $\emptyset(x,y,t,q)$ on R^n belongs to $E(R^n)$, if for each compact set, $J \subset S_{c,d}$ where

$S_{a,b} = \{x,y: x,y\epsilon R^n, |x| \le a, |y| \le b, a > 0, b > 0\}$
$S_{c,d} = \{t,q: t,q\epsilon R^n, |t| \le c, |q| \le d, c > 0, d > 0\}$

$$\gamma_{E,m,n,k,l}[\emptyset(x,y,t,q)] = \sup_{\substack{x,y\epsilon I\\ t,q\epsilon J}} \left|D^{m,n,k,l}_{x,y,t,q}\emptyset(x,y,t,q)\right| < \infty \text{-- (3)}$$

Thus $E(R^n)$ will denote the space of all $\emptyset(x,y,t,q)\epsilon E(R^n)$ with compact support contained in $S_{a,b} \cap S_{c,d}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y,t,q)$ is a fractional Fourier-Mellin transformable if it is a member of E.

1.3 Distributional Two Dimensional Fractional Fourier-Mellin Transform (2DFRFMT)

The two dimensional distributional Fractional Fourier -Mellin transform of $f(x,y,t,q)\epsilon E^*(R^n)$ can be defined by

$$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle \text{--------------------- (4)}$$

where, $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1} q^{\frac{2\pi i\chi}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$

$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}\, t^{C_{1\theta}i\lambda-1} q^{C_{1\theta}i\chi-1} e^{C_{1\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$, $C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ $0 < \alpha < \frac{\pi}{2}, 0 < \theta < \frac{\pi}{2}$. ---------------------- (5)

Right hand side of equation (4) has a meaning as the application of $f(x,y,t,q)\epsilon E^*(R^n)$ to $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\epsilon E$.
It can be extended to the complex space as an entire function given by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi',\eta',\lambda',\chi')$
$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\rangle$-- (6)

The right hand side is meaningful because for each $\xi',\eta',\lambda',\chi'\epsilon C^n$, $K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\epsilon E$ as a function of x,y,t,q.
Motivated by the above work, we have generalized two dimensional fractional Fourier-Mellin transform in the distributional sense. In this paper we proposed some operational calculus such as shifting operator, scaling operator and shifting-scaling operator for two dimensional fractional Fourier-Mellin transform.

Results

2.1] Shifting operator

If $\emptyset(x,y,t,q)\epsilon E$ and τ & δ are real numbers then $\emptyset(x+\tau, y+\delta, t, q)\epsilon E$, $x+\tau > 0, y+\delta > 0$
where $\emptyset(x,y,t,q) = K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$.
Proof:-
Consider,

$$\gamma_{E,m,n,l,j}\emptyset(x+\tau,y+\delta,t,q) = \sup_{\substack{x,y\epsilon I\\ t,q\epsilon I}} \left|D^{m,n,l,j}_{x,y,t,q}\emptyset(x+\tau,y+\delta,t,q)\right|$$

$$= \sup_{\substack{x,y\epsilon I\\ t,q\epsilon I}} \left|D^{m,n,l,j}_{x,y,t,q} K_{\alpha,\theta}(x+\tau,y+\delta,t,q,\xi,\eta,\lambda,\chi)\right|$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n} C_{m,n}C_\alpha[(x+\tau)cos\alpha - \xi]^{m-2r}\,[(y+\delta)cos\alpha - \eta]^{n-2s}\, e^{(m-r)u+(n-s)v}$$

$$\sum_{k=0}^{l}\sum_{h=0}^{j} l!\,j!\left(\frac{2\pi i}{tan\theta}\right)^{l-k+j-h} C_k(t)C_k(q)(logt)^{l-k}(logq)^{h-j}e^{(l-k)w+(j-h)z}$$

$$\left(\frac{1}{t}\right)^{2l-k}\left(\frac{1}{q}\right)^{2h-j} P(\lambda)P(\chi)t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}$$

$< \infty$

where, $C_{m,n} = \frac{m!}{r!(m-2r)!}\frac{n!}{s!(n-2s)!}(2i)^{m-r+n-s}$

$C_\alpha = cos\alpha^{r+s}(C_{2\alpha})^{m-r+n-s}$
$u = iC_{2\alpha}\{[(x+\tau)^2+\xi^2]cos\alpha - 2(x+\tau)\xi\}$
$v = iC_{2\alpha}\{[(y+\delta)^2+\eta^2]cos\alpha - 2(y+\delta)\eta\}$

$C_k(t) = \frac{l!}{(l-2k)!}\left[\frac{1-logt}{2tlogt}\right]^k$,

$$C_k(q) = \frac{j!}{(j-2h)!}\left[\frac{1-logq}{2qlogq}\right]^h$$

For any x' & y',u,$v \in R^n$ and any fixed p & q, $0 < \alpha \leq \frac{\pi}{2}$, $0 < \theta \leq \frac{\pi}{2}$.
Thus $\emptyset(x+\tau, y+\delta, t, q) \in E$, $x+\tau > 0$, $y+\delta > 0$.

2] Scaling operator

If $\emptyset(x,y,t,q) \epsilon E$ and $\mu > 0$ & $\rho > 0$ then scaling operator $R: E \to E$ defined by $R\emptyset = \varphi$ where, $\frac{1}{\mu\rho}\varphi(x,y,t,q) = \emptyset(x,y,\mu t,\rho q)$ is a topological automorphism.

Proof:-

Consider,

$$\gamma_{E,m,n,l,j}\emptyset\left(x,y,\frac{t}{\mu},\frac{q}{\rho}\right) = \sup_{\substack{x,y \in I \\ t,q \in I}}\left|D_{x,y,t,q}^{m,n,l,j}\left\{\frac{1}{\mu\rho}\emptyset\left(x,y,\frac{t}{\mu},\frac{q}{\rho}\right)\right\}\right|$$

$$= \sup_{\substack{x,y \in I \\ t,q \in I}}\left|D_{x,y,t,q}^{m,n,l,j}\left\{K_{\alpha,\theta}\left(x,y,\frac{t}{\mu},\frac{q}{\rho},\xi,\eta,\lambda,\chi\right)\right\}\right|$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n} C_{m,n}C_\alpha[x\cos\alpha - \xi]^{m-2r}\,[y\cos\alpha - \eta]^{n-2s}\, e^{(m-r)u+(n-s)v}$$

$$\sum_{k=0}^{l}\sum_{h=0}^{j} l!\,j!\left(\frac{2\pi i}{\tan\theta}\right)^{l-k+j-h} C_k\left(\frac{t}{\mu}\right)C_k\left(\frac{q}{\rho}\right)\left(\log\frac{t}{\mu}\right)^{l-k}\left(\log\frac{q}{\rho}\right)^{h-j} e^{(l-k)w+(j-h)z}$$

$$\left(\frac{\mu}{t}\right)^{2l-k}\left(\frac{\rho}{q}\right)^{2h-j} P(\lambda)P(\chi)t^{\frac{2\pi i\lambda}{\sin\theta}-1}q^{\frac{2\pi i\chi}{\sin\theta}-1}$$

where, $C_{m,n} = \frac{m!}{r!(m-2r)!}\frac{n!}{s!(n-2s)!}(2i)^{m-r+n-s}$

$C_\alpha = \cos\alpha^{r+s}(C_{2\alpha})^{m-r+n-s}$

$u = iC_{2\alpha}\{[(x)^2+\xi^2]\cos\alpha - 2(x)\xi\}$

$v = iC_{2\alpha}\{[(y)^2+\eta^2]\cos\alpha - 2(y)\eta\}$

$$C_k\left(\frac{t}{\mu}\right) = \frac{l!}{(l-2k)!}\left[\frac{1-\log\frac{t}{\mu}}{2\frac{t}{\mu}\log\frac{t}{\mu}}\right]^k,$$

$$C_k\left(\frac{q}{\rho}\right) = \frac{j!}{(j-2h)!}\left[\frac{1-\log\frac{q}{\rho}}{2\frac{q}{\rho}\log\frac{q}{\rho}}\right]^h$$

$< \infty$

For any x' & y',u,$v \in R^n$ and any fixed p & q, $0 < \alpha \leq \frac{\pi}{2}$, $0 < \theta \leq \frac{\pi}{2}$.
Thus $\emptyset\left(x,y,\frac{t}{\mu},\frac{q}{\rho}\right) \in E$, $x+\tau > 0$, $y+\delta > 0$.

3] Shifting Scaling Operator

If $\emptyset(x,y,t,q) \epsilon E$ and τ, δ, μ & ρ are real numbers then $\emptyset\left(x+\tau, y+\delta, \frac{t}{\mu},\frac{q}{\rho}\right) \epsilon E$, $x+\tau > 0$, $y+\delta > 0$, $\frac{t}{\mu} > 0$ & $\frac{q}{\rho} > 0$ then scaling operator $R: E \to E$ defined by $R\emptyset = \varphi$
where, $\frac{1}{\mu\rho}\varphi(x,y,t,q) = \emptyset(x+\tau, y+\delta, \mu t, \rho q)$ is a topological automorphism.

Proof:-

Consider,

$$\gamma_{E,m,n,l,j}\emptyset\left(x,y,\frac{t}{\mu},\frac{q}{\rho}\right) = \sup_{\substack{x,y \in I \\ t,q \in I}}\left|D_{x,y,t,q}^{m,n,l,j}\left\{\frac{1}{\mu\rho}\emptyset\left(x+\tau,y+\delta,\frac{t}{\mu},\frac{q}{\rho}\right)\right\}\right|$$

$$= \sup_{\substack{x,y \in I \\ t,q \in I}}\left|D_{x,y,t,q}^{m,n,l,j}\left\{K_{\alpha,\theta}\left(x+\tau,y+\delta,\frac{t}{\mu},\frac{q}{\rho},\xi,\eta,\lambda,\chi\right)\right\}\right|$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n} C_{m,n}C_\alpha[(x+\tau)\cos\alpha - \xi]^{m-2r}\,[(y+\delta)\cos\alpha - \eta]^{n-2s}\, e^{(m-r)u+(n-s)v}$$

$$\sum_{k=0}^{l}\sum_{h=0}^{j} l!\,j!\left(\frac{2\pi i}{\tan\theta}\right)^{l-k+j-h} C_k\left(\frac{t}{\mu}\right)C_k\left(\frac{q}{\rho}\right)\left(\log\frac{t}{\mu}\right)^{l-k}\left(\log\frac{q}{\rho}\right)^{h-j} e^{(l-k)w+(j-h)z}$$

$$\left(\frac{\mu}{t}\right)^{2l-k}\left(\frac{\rho}{q}\right)^{2h-j} P(\lambda)P(\chi)t^{\frac{2\pi i\lambda}{\sin\theta}-1}q^{\frac{2\pi i\chi}{\sin\theta}-1}$$

where, $C_{m,n} = \frac{m!}{r!(m-2r)!}\frac{n!}{s!(n-2s)!}(2i)^{m-r+n-s}$

$C_\alpha = \cos\alpha^{r+s}(C_{2\alpha})^{m-r+n-s}$

$u = iC_{2\alpha}\{[(x+\tau)^2+\xi^2]\cos\alpha - 2(x+\tau)\xi\}$

$v = iC_{2\alpha}\{[(y+\delta)^2+\eta^2]\cos\alpha - 2(y+\delta)\eta\}$

$$C_k\left(\frac{t}{\mu}\right) = \frac{l!}{(l-2k)!}\left[\frac{1-\log\frac{t}{\mu}}{2\frac{t}{\mu}\log\frac{t}{\mu}}\right]^k,$$

$$C_k\left(\frac{q}{\rho}\right) = \frac{j!}{(j-2h)!}\left[\frac{1-log\frac{q}{\rho}}{2\frac{q}{\rho}log\frac{q}{\rho}}\right]^h$$

$< \infty$

For any x' & y',$u,v \in R^n$ and any fixed p & q, $0 < \alpha \leq \frac{\pi}{2}, 0 < \theta \leq \frac{\pi}{2}$.

Thus $\emptyset\left(x+\tau, y+\delta, \frac{t}{\mu}, \frac{q}{\rho}\right) \in E$, $x+\tau > 0, y+\delta > 0$.

Operators on the space E and its Dual space E*.

3.1 Theorem- The operator $\emptyset(x,y,t,q) \to D_{x,y}[\emptyset(x,y,t,q)]$ is defined on the space E and transform this space E into itself, where $\emptyset(x,y,t,q) = K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$.

Proof-

Let, $D_{x,y}[\emptyset(x,y,t,q)] = \emptyset_1(x,y,t,q)$,

We have,

$$\gamma_{E,m,n,l,j}\emptyset(x,y,t,q) = \sup_{\substack{x,y \in I\\ t,q \in I}}\left|D^{m,n,l,j}_{x,y,t,q}\emptyset_1(x,y,t,q)\right|$$

$$= \sup_{\substack{x,y \in I\\ t,q \in I}}\left|D^{m,n,l,j}_{x,y,t,q}\{D_{x,y}[\emptyset(x,y,t,q)]\}\right|$$

$$= \sup_{\substack{x,y \in I\\ t,q \in I}}\left|D^{m+1,n+1,l,j}_{x,y,t,q}\emptyset(x,y,t,q)\right|$$

$$= \sup_{\substack{x,y \in I\\ t,q \in I}}\left|D^{m+1,n+1,l,j}_{x,y,t,q}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\right|$$

$$= \sup_{\substack{x,y \in I\\ t,q \in I}}\left|D^{m+1,n+1,l,j}_{x,y,t,q}\left\{\begin{matrix}\sqrt{\frac{1-icot\alpha}{2\pi}}\,e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\\ t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}\end{matrix}\right\}\right|$$

$$= \sum_{r=0}^{m+1}\sum_{s=0}^{n+1} C_{m+1,n+1}C_\alpha(xcos\alpha-\xi)^{m+1-2r}(ycos\alpha-\eta)^{n+1-2s}$$

$$e^{(m-r+1)u+(n-s+1)v}\sum_{k=0}^{l}\sum_{h=0}^{j} l!\,j!\left(\frac{2\pi i}{tan\theta}\right)^{l-k+j-h}C_k(t)C_h(q)$$

$$(\log t)^{l-k}(\log q)^{j-h}e^{(l-k)w-(j-h)z}\left(\frac{1}{t}\right)^{2l-k}\left(\frac{1}{q}\right)^{2j-h}P(\lambda)P(\chi)t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}$$

$$< \infty$$

where,

$$C_{m+1,n+1} = \frac{(m+1)!}{r!\,(m+1-2r)!}\frac{(n+1)!}{s!\,(n+1-2s)!}(2i)^{m-r+n-s+2}$$

$$C_\alpha = (cos\alpha)^{r+s}(C_{2\alpha})^{m-r+n-s+2}$$

$$u = iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]$$

$$v = iC_{2\alpha}[(y^2+\eta^2)cos\alpha-2y\eta]$$

$$w = e^{\frac{\pi i}{tan\theta}[log^2t]}$$

$$z = e^{\frac{\pi i}{tan\theta}[log^2q]}$$

$$C_k(t) = \frac{l!}{(l-2k)!}\left(\frac{1-log\,t}{2tlogt}\right)^k$$

$$C_h(q) = \frac{j!}{(j-2h)!}\left(\frac{1-log\,q}{2qlogq}\right)^h$$

For any $x,y,t,q \in E(R^n)$ and any fixed m,n,,j, $0 < \alpha \leq \frac{\pi}{2}, 0 < \theta \leq \frac{\pi}{2}$.

$\therefore \emptyset_1(x,y,t,q) \in E$

$\therefore D_{x,y}[\emptyset(x,y,t,q)] \in E$.

3.2 Theorem- The operator $\emptyset(x,y,t,q) \to D_{t,q}[\emptyset(x,y,t,q)]$ is defined on the space E and transform this space E into itself, where $\emptyset(x,y,t,q) = K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$.

Proof-

Let, $D_{t,q}[\emptyset(x,y,t,q)] = \emptyset_1(x,y,t,q)$, We have,

$$\gamma_{E,m,n,l,j}\emptyset(x,y,t,q) = \sup_{\substack{x,y\in I\\ t,q\in I}}\left|D^{m,n,l,j}_{x,y,t,q}\emptyset_1(x,y,t,q)\right|$$

$$= \sup_{\substack{x,y\in I\\ t,q\in I}}\left|D^{m,n,l,j}_{x,y,t,q}\{D_{t,q}[\emptyset(x,y,t,q)]\}\right|$$

$$= \sup_{\substack{x,y\in I\\ t,q\in I}}\left|D^{m,n,l+1,j+1}_{x,y,t,q}\emptyset(x,y,t,q)\right|$$

$$= \sup_{\substack{x,y\in I\\ t,q\in I}}\left|D^{m,n,l+1,j+1}_{x,y,t,q}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\right|$$

$$= \sup_{\substack{x,y\in I\\ t,q\in I}}\left|D^{m,n,l+1,j+1}_{x,y,t,q}\left\{\begin{matrix}\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\\ t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}\end{matrix}\right\}\right|$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n} C_{m,n}C_\alpha(xcos\alpha-\xi)^{m-2r}(ycos\alpha-\eta)^{n-2s}$$

$$e^{(m-r)u+(n-s)v}\sum_{k=0}^{l+1}\sum_{h=0}^{j+1}(l+1)!\,(j+1)!\,\left(\frac{2\pi i}{tan\theta}\right)^{l-k+j-h+2}C_k(t)C_h(q)$$

$$(logt)^{l-k+1}(logq)^{j-h+1}e^{(l-k+1)w-(j-h+1)z}\left(\frac{1}{t}\right)^{2l-k+1}\left(\frac{1}{q}\right)^{2j-h+1}P(\lambda)P(\chi)t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}$$

$$< \infty$$

where,

$$C_{m,n} = \frac{m!}{r!\,(m-2r)!}\,\frac{n!}{s!\,(n-2s)!}(2i)^{m-r+n-s}$$

$$C_\alpha = (cos\alpha)^{r+s}(C_{2\alpha})^{m-r+n-s}$$

$$u = iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]$$

$$v = iC_{2\alpha}[(y^2+\eta^2)cos\alpha-2y\eta]$$

$$w = e^{\frac{\pi i}{tan\theta}[log^2t]}$$

$$z = e^{\frac{\pi i}{tan\theta}[log^2q]}$$

$$C_k(t) = \frac{l!}{(l-2k)!}\left(\frac{1-log\,t}{2tlogt}\right)^k$$

$$C_h(q) = \frac{j!}{(j-2h)!}\left(\frac{1-log\,q}{2qlogq}\right)^h$$

For any $x,y,t,q\in E(R^n)$ and any fixed m,n,l,j, $0<\alpha\le\frac{\pi}{2}$, $0<\theta\le\frac{\pi}{2}$.

$\therefore\ \emptyset_1(x,y,t,q)\in E$

$\therefore\ D_{t,q}[\emptyset(x,y,t,q)]\in E.$

Conclusion

In this paper we discussed on shifting operator, scaling operator and shifting-scaling operator. of two dimensional Fourier-Mellin transform.

References

1. Derrode S, Ghorbel F. Robust and Efficient Fourier-Mellin Transform Approximations for Gray-Level Image Reconstruction and Complete Invariant Description", Computer Vision and Image Understanding 2001; 83(1):57.
2. Richard Altes A. The Fourier–Mellin transform and mammalian hearing, J Acoustic Society of America. 1978; 63:174.
3. Carlo Bardaro y Paul L. Butzer z Ilaria Mantellini. The foundations of fractional Mellin transform analysis arXiv:1406.6202v1 [math.FA] 24 Jun, 2014.
4. Sharma VD, Deshmukh PB. S-Type Spaces For Two Dimensional Fractional Fourier-Mellin Transform International Journal of Advances in Science Engineering and Technology, ISSN: 2321-9009 Special 2015, (1).
5. Martinez-de Dios JR. Ollero A. A Real-Time Image Stabilization System Based on Fourier-Mellin Transform, Lecture Notes in Computer Science, 3211, 376-383.
6. Lin CY, Wu M, Bloom JA, Cox M, Miller LJ, Lui YM. Rotation, Scale and Translation Resilient Watermarking for Images, IEEE Transactions on Image Processing, 2001; 10(5):767-782.
7. Moller R, Salguero H, Salguero E. Image Recognition Using the Fourier-Mellin Transform, LIPSE-SEPI-ESIME-IPN, Mexico, 2004.

8. Page GS. An Investigation of Techniques in Deformable Object Recognition, Rochester Institute of Technology Rochester, New York.
9. Pratt JG. Application of the Fourier-Mellin Transform to Translation, Rotation and Scale Invariant Plant Leaf Identi cation. Montreal, 2000.
10. Sharma VD, Deshmukh PB. Operation transform formulae for two dimensional fractional Mellin transform, International Journal of Science & Research 2014, 3(9).
11. Sharma VD, Deshmukh PB. Operational Calculus on Two Dimensional Fractional Mellin Transform, International Journal of Mathematical Archieve. 2014; 5(9):242-246.