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Common fixed point theorem for weakly compatible mappings in complex valued metric spaces using identity function

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Abstract
The notion of complex-valued metric spaces was introduced by Azam *et al.* He introduced the new concept and established a common fixed point result in the context of complex-valued metric spaces. In this paper, we have tried to prove a common fixed point theorem for weakly compatible mappings in complex valued metric spaces taking an identity function. Our results generalize some recent result in the literature due to Azam and Sintunavarat.

Keywords: complex valued metric space, point of coincidence, Cauchy and convergent sequence, weakly compatible mappings, common fixed point

1. Introduction

It is well known fact that the mathematical results regarding fixed points of contraction-type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models. The theory of fixed points has been developed, regarding the results to finding the fixed points self and nonself over the last 50 years. Fixed point theorems have expensive applications in proving the existence and uniqueness of the solutions of differential equations, integral equations, partial differential equations and in other related areas. Banach's fixed point theorem plays a major role in fixed point theory. It has application in many branches of Mathematics. Because of its usefulness, a lot of article has been dedicated to the improvement and generalization of that result. Most of these generalizations have been made by considering different contractive type conditions in different spaces. In 2011, Azam [1] made one such generalization by introducing a complex valued metric space. Very recently, Sintunavarat [10] generalized this result by replacing the constant of contraction by some control functions. The purpose of this work is to obtain a common fixed point result for weakly compatible mappings in complex valued metric spaces which generalizes the result of [1] and [10].

2. Preliminaries

Let $\mathbb{C}$ be the set of complex number and let $z_1, z_2 \in \mathbb{C}$. We can define a partial ordering $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ iff $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.
It follows that, $z_1 \precsim z_2$ if one of the following conditions is satisfied:

1. $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
2. $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
3. $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;
4. $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \ne z_2$ and one of (ii), (iii), and (iv) is satisfied and we will write $z_1 \prec z_2$ if only (iv) is satisfied. Note that

1. $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| \le |z_2|$;
2. $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$;
3. $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$;
4. $a, b \in R$, $0 \le a \le b$ and $z_1 \precsim z_2 \Rightarrow az_1 \precsim bz_2$.
5.

Definition 2.1. [1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

1. $0 \preceq d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$;
2. $d(x,y) = d(y,x)$, for all $x, y \in X$
3. $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space. Note that $d(x,y) \preceq 1 + d(x,y)$ and so, $|\frac{d(x,y)}{1+d(x,y)}| \leq 1$.

Example 2.2. [1] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$, where $k \in R$. Then (X,d) is a complex valued metric space.

Definition 2.3. [1] Let (X,d) be a complex valued metric space, (x_n) be a sequence in X and $x \in X$.

1. If for every c ϵ ℂ, with 0 ≺ c there is $n_0 \in N$ such that for all $n > n_0$, $d(x,y) \prec$ c, then (x_n) is said to be convergent, (x_n) converges to x and x is the limit point of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. If for every c ϵ ℂ, with 0 ≺ c there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec$ c, where $m \in N$, then (x_n) is said to be Cauchy sequence.
3. If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued metric space.

Lemma 2.4. [1] Let (X,d) be a complex valued metric space and let (x_n) be a sequence in X . then (x_n) converges to x iff $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 2.5. [1] Let (X,d) be complex valued metric space and let (x_n) be a sequence in X . then (x_n) is a Cauchy sequence iff $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.

Definition 2.6. [7] Let T and S be self-mapping of a non-empty set X . The mapping T and S are weakly compatible if $TSx = STx$ whenever $Tx = Sx$.

Definition 2.7. Let (X,d) be a complex valued metric space. A mapping $T : X \to X$ is said to be contractive if $d(Tx,Ty) \preceq d(x,y)$ for all $x, y \in X$.

3. Main Results

In this section, we always suppose that ℂ is the set of complex numbers and $\preceq$ is a partial ordering on ℂ. Throughout the paper we denote by N the set of natural numbers.

Lemma 3.1. [2] Let X be a non-empty set and the mappings S , T :, $I : X \to X$ have a unique point of coincidence v in X . If (S,I) and (T,I) are weakly compatible, then S , T and I have a unique common fixed point.

Theorem. [10] Let (X,d) be a complete complex valued metric space and S , $T : X \to X$.

Suppose there exist mappings $g_1, g_2 : X \to [0,1)$ such that for all $x, y \in X$.

1. $g_i(Sx) \leq g_i(x)$ and $g_i(Tx) \leq g_i(x)$ for $i = 1,2$;
2. $g_1(x) + g_2(x) < 1$;
3. $d(Sx,Ty) \preceq g_1(x)d(x,y) + \frac{g_2(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}$

Then S and T have a unique common fixed point in X .

Proof: Let $x_0 \in X$ be an arbitrary point in X . Choose a point $x_1 \in X$ such that $Ix_1 = Sx_0$ which is possible since $S(X) \subseteq I(X)$. Also, we may choose a point $x_2 \in X$ satisfying $Ix_2 = Tx_1$ since $T(X) \subseteq I(X)$. Continuing in this way, we can construct a sequence (Ix_n) in $I(X)$ such that

$Ix_0 = Sx_{n-1}$, if n is odd
$= Tx_{n-1}$, if n is even.

If $n \in N$ is odd, then by using hypothesis we obtain $d(Ix_n, Ix_{n+1}) = d(Sx_{n-1}, Tx_n)$

$$\leq g_1(Ix_{n-1})d(Ix_{n-1}, Ix_n) + \frac{g_2(Ix_{n-1})d(Ix_{n-1}, Sx_{n-1})d(Ix_n, Tx_n)}{1 + d(Ix_{n-1}, Ix_n)}$$

$$= g_1(Ix_{n-1})d(Ix_{n-1}, Ix_n) + \frac{g_2(Ix_{n-1})d(Ix_{n-1}, Ix_n)d(Ix_n, Tx_{n+1})}{1 + d(Ix_{n-1}, Ix_n)}$$

Therefore,

$$|d(Ix_n, Ix_{n+1})| \leq g_1(Ix_{n-1})|d(Ix_{n-1}, Ix_n)| + g_2(Ix_{n-1})|d(Ix_n, Ix_{n+1})| \left| \frac{d(Ix_n, Ix_{n+1})}{1 + d(Ix_{n-1}, fx_n)} \right|$$

$$\leq g_1(Ix_{n-1})|d(Ix_{n-1}, Ix_n)| + g_2(Ix_{n-1})|d(Ix_n, Ix_{n+1})|$$

$$= g_1(Tx_{n-2})|d(Ix_{n-1}, Ix_n)| + g_2(Tx_{n-2})|d(Ix_n, Ix_{n+1})|$$

$$\leq g_1(Ix_{n-2})|d(Ix_{n-1}, Ix_n)| + g_2(Ix_{n-2})|d(Ix_n, Ix_{n+1})|$$

$$= g_1(Sx_{n-3})|d(Ix_{n-1}, Ix_n)| + g_2(Sx_{n-3})|d(Ix_n, Ix_{n+1})|$$

$$\leq g_1(Ix_{n-3})|d(Ix_{n-1}, Ix_n)| + g_2(Ix_{n-3})|d(Ix_n, Ix_{n+1})|$$

$$\leq g_1(Ix_0)|d(Ix_{n-1}, Ix_n)| + g_2(Ix_0)|d(Ix_n, Ix_{n+1})|$$

Which implies that $|d(Ix_n, Ix_{n+1})| \leq \dfrac{g_1(Ix_0)}{1 - g_2(Ix_0)} |d(Ix_{n-1}, Ix_n)|$

If $n \in N$ is even, then

$$d(Ix_n, Ix_{n+1}) = d(Tx_{n-1}, Sx_n) = d(Sx_n, Tx_{n-1})$$

$$\leq g_1(Ix_n)d(Ix_n, Ix_{n-1}) + \frac{g_2(Ix_n)d(Ix_n, Sx_n)d(Ix_{n-1}, Tx_{n-1})}{1 + d(Ix_n, Ix_{n-1})}$$

$$= g_1(Ix_n)d(Ix_n, Ix_{n-1}) + \frac{g_2(Ix_n)d(Ix_n, Ix_{n+1})d(Ix_{n-1}, Ix_n)}{1 + d(Ix_n, Ix_{n-1})}$$

Therefore,

$$|d(Ix_n, Ix_{n+1})| \leq g_1(Ix_n)|d(Ix_n, Ix_{n-1})| + g_2(Ix_n)|d(Ix_n, Ix_{n+1})| \left| \frac{d(Ix_{n-1}, Ix_n)}{1 + d(Ix_n, Ix_{n-1})} \right|$$

$$\leq g_1(Ix_n)d(Ix_n, Ix_{n-1}) + g_2(Ix_n)d(Ix_n, Ix_{n+1})$$

$$= g_1(Tx_n)d(Ix_n, Ix_{n-1}) + g_2(Tx_n)d(Ix_n, Ix_{n+1})$$

$$\leq g_1(Ix_{n-1})|d(Ix_n, Ix_{n-1})| + g_2(Ix_{n-1})|d(Ix_n, Ix_{n+1})|$$

$$= g_1(Sx_{n-2})|d(Ix_n, Ix_{n-1})| + g_2(Sx_{n-2})|d(Ix_n, Ix_{n+1})|$$

$$\leq g_1(Ix_0)|d(Ix_n, Ix_{n-1})| + g_2(fx_0)|d(Ix_n, Ix_{n+1})|$$

Which gives that

$$|d(Ix_n, Ix_{n+1})| \leq \frac{g_1(Ix_0)}{1-g_2(Ix_0)}|d(Ix_n, Ix_{n-1})|$$

Thus for any positive integer n, it must be the case that

$$|d(Ix_n, Ix_{n+1})| \leq \frac{g_1(Ix_0)}{1-g_2(Ix_0)}|d(Ix_{n-1}, Ix_n)| \tag{3.1}$$

If we let $\alpha = \frac{g_1(Ix_0)}{1-g_2(Ix_0)}$, then by repeated application of (3.1)

$$|d(Ix_n, Ix_{n+1})| \leq \alpha|d(Ix_{n-1}, Ix_n)|$$

$$\leq \alpha^2|d(Ix_{n-2}, Ix_{n-1})|$$

$$\leq \alpha^n|d(Ix_0, Ix_1)|$$

Now, for all m, $n \in N$, $m > n$, we have

$$d(Ix_n, Ix_m) \preceq d(Ix_n, Ix_{n+1}) + d(Ix_{n+1}, Ix_{n+2}) + \ldots\ldots\ldots\ldots + d(Ix_{m-1}, Ix_m).$$

Therefore,

$$d(Ix_n, Ix_m) \leq |d(Ix_n, Ix_{n+1})| + |d(Ix_{n+1}, Ix_{n+2})| + \ldots\ldots\ldots + |d(Ix_{m-1}, Ix_m)|$$

$$\leq (\alpha^n + \alpha^{n+1} + \ldots\ldots\ldots + \alpha^{m-1})|d(Ix_{m-1}, Ix_m)|$$

$$\leq \frac{\alpha^n}{1-\alpha}|d(Ix_0, Ix_1)|$$

Since $\alpha \in [0,1)$ taking limit as $m, n \to \infty$ we have $|d(Ix_n, Ix_m)| \to 0$ which implies that (Ix_n) is a Cauchy sequence in $I(X)$. By completeness of $I(X)$, there exist u, $v \in X$ such that
$Ix_n \to v = Iu$

Now,

$$d(Iu, Tu) \preceq d(Iu, Ix_{2n+1}) + d(Ix_{2n+1}, Tu)$$

$$= d(Iu, Ix_{2n+1}) + d(Sx_{2n}, Tu)$$

$$\preceq d(Iu, Ix_{2n+1}) + g_1(Ix_{2n})d(Ix_{2n}, Iu) + \frac{g_2(Ix_{2n})d(Ix_{2n}, Sx_{2n})d(Iu, Tu)}{1 + d(Ix_{2n}, Iu)}$$

Which implies that

$$|d(Iu, Tu)| \leq d(Iu, Ix_{2n+1}) + g_1(Ix_{2n})d(Ix_{2n}, Iu) + \frac{g_2(Ix_{2n})d(Ix_{2n}, Sx_{2n})d(Iu, Tu)}{1 + d(Ix_{2n}, Iu)}$$

$$\leq |d(Iu, Ix_{2n+1})| + g_1(Ix_{2n})|d(Ix_{2n}, Iu)| + g_2(Ix_{2n})|d(Ix_{2n}, Sx_{2n})||d(Iu, Tu)|,$$

Since $1 \leq 1+ d(Ix_{2n}, Iu)$

$$\leq |d(Iu, Ix_{2n+1})| + g_1(Ix_0)|d(Ix_{2n}, Iu)| + g_2(Ix_0)|d(Ix_{2n}, Ix_{2n+1})||d(Iu, Tu)|$$

Taking $n \to \infty$, it follows that $|d(Iu, Tu)| = 0$ and hence $d(Iu, Tu) = 0$.

Therefore, $Iu = Tu = v$. Similarly, we can show that $Iu = Su = v$.

Thus, $Iu = Su = Tu = v$ and so v becomes a common point of coincidence of I, S and T.
For uniqueness, let there exists another point $w(\neq v) \in X$ such that $Ix = Sx = Tx = w$ for some $x \in X$. Thus,

$$\begin{aligned} d(v, w) &= d(Su, Tx) \\ &\leq g_1(Iu)d(Iu, Ix) + \frac{g_2(Iu)d(Iu, Su)d(Ix, Tx)}{1 + d(Iu, Ix)} \\ &= g_1(v)d(v, w) + \frac{g_2(v)d(v, v)d(w, w)}{1 + d(v, w)} \\ &= g_1(v)d(v, w) \end{aligned}$$

Which implies that

$$|d(v, w)| \leq g_1(v)|d(v, w)|$$

Since $0 \leq g_1(v) < 1$, it follows that $|d(v, w)| = 0$ and so $v = w$. If (S, I) and (T, I) are weakly compatible, then by Lemma (3.1), I, S and T have a unique common fixed point in X.

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