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# Observation on the Biquadratic Equation with Five Unknowns $(x-y)(x^3+y^3)=(2k^2+2k+2)(X^2-Y^2)w^2$

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**Abstract**

We obtain infinitely many non-zero integer quintuples $(x, y, X, Y, w)$ satisfying the Bi-Quadratic equation $(x-y)(x^3+y^3)=(2k^2+2k+2)(X^2-Y^2)w^2$. Various interesting relations between the solutions and special numbers, namely, Polygonal numbers, Pronic numbers, Pyramidal numbers, Stella Octangular numbers, and Octahedral numbers are exhibited.

**Keywords:** Biquadratic equation with five unknowns, Integral solutions, Polygonal and Pyramidal numbers.
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**Notations Used**

$t_{m,n}$ : Polygonal number of rank n with sides m

$$t_{m,n} = n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

$P_n^m$ : Pyramidal number of rank n with sides m

$$P_n^m = \frac{n\ (n+1)}{6}\left[(m-2)n+(5-m)\right]$$

$SO_n$ : Stella Octangular number of rank n

$$SO_n = n(2n^2-1)$$

$Pr_n$ : Pronic number of rank n

$$Pr_n = n(n+1)$$

$OH_n$ : Octahedral number of rank n

$$OH_n = \frac{1}{3}\left[n(2n^2+1)\right]$$

## 1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. In this context one may refer [5-10, 12, 13] for various problems on the biquadratic Diophantine equations with four and five variables [11]. have discussed the problem on biquadratic Diophantine equation with five unknowns. This paper concerns with yet another problem of determining non-trivial integral solutions of the non-homogeneous biquadratic equation with five unknowns given by $(x-y)(x^3+y^3)=(2k^2+2k+2)(X^2-Y^2)w^2$. A few relations among the solutions are presented.

## 2. Method of Analysis

The Diophantine equation representing the biquadratic equation with five unknowns under consideration is

$$(x - y)(x^3 + y^3) = (2k^2 + 2k + 2)(X^2 - Y^2)w^2 \quad (1)$$

Introducing the linear transformations

$$x = u + v, \quad X = 2u + v, \quad y = u - v, \quad Y = 2u - v \quad (2)$$

in (1), it simplifies to

$$u^2 + 3v^2 = (4k^2 + 4k + 4)w^2 \quad (3)$$

The above equation (3) is solved through different choices and thus, one obtains distinct patterns of integer solutions to (1)

### 2.1 Pattern: 1

Let $w = a^2 + 3b^2$ (4)

Substituting (4) in (3) and using the method of factorization, define

$$(u + i\sqrt{3}v) = (a + i\sqrt{3}b)^2((2k+1) + i\sqrt{3})$$

Equating real and imaginary parts, we have

$$u = a^2 - 3b^2 + 2ka^2 - 6kb^2 - 6ab$$
$$v = a^2 - 3b^2 + 2ab + 4kab$$

Substituting the values of u and v in (2), the non-zero distinct integral solutions of (1) are given by

$$x(a,b) = 2a^2 - 6b^2 + 2ka^2 - 6kb^2 - 4ab + 4kab$$
$$y(a,b) = 2ka^2 - 6kb^2 - 8ab - 4kab$$
$$X(a,b) = 3a^2 - 9b^2 + 4ka^2 - 12kb^2 - 10ab + 4kab \quad (5)$$
$$Y(a,b) = a^2 - 3b^2 + 4ka^2 - 12kb^2 - 14ab - 4kab$$
$$w(a,b) = a^2 + 3b^2$$

### 2.2 Properties

$$X(a,a+1) - Y(a,a+1) + 4t_{4,a} + 4Pr_a(-1-2k) + 12a + 6 \equiv 0$$
$$x(a,2a-1) - y(a,2a-1) + t_{46,a} - 4t_{6,a}(1+2k) - 3a + 6 \equiv 0$$
$$X(a,3a-1) + 3w(a,3a-1) + 2t_{4,a}(-3+52k) - 4t_{5,a}(2k-5) \equiv 12 \pmod{72}$$
$$Y(a,2a^2-1) - w(a,2a^2-1) - 2t_{4,a}(1+26k) + 4k(12t_{4,a^2} + 3) + 2SO_a[2k+7] \equiv 0$$
$$x(a^2,a+1) - X(a^2,a+1) - 2(2k+1)t_{4,a} - 12P_a^5 \equiv 0 \pmod{2k+1}$$

### 2.3 Pattern: 2

(3) can be written as

$$u^2 + 3v^2 = (4k^2 + 4k + 4)w^2 * 1 \quad (6)$$

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (7)$$

Using (7) in (6) and employing the method of factorization, define

$$(u + i\sqrt{3}v) = ((2k+1) + i\sqrt{3})(a + i\sqrt{3}b)^2 * \frac{(1+i\sqrt{3})}{2} \quad (8)$$

Equating real and imaginary parts in (8), we get

$$u = -a^2 + 3b^2 - 6ab + ka^2 - 3kb^2 - 6kab$$
$$v = a^2 - 3b^2 - 2ab + ka^2 - 3kb^2 + 2kab \quad (9)$$

Using (9) and (2), we get the integral solution of (1) as

$$x(a,b) = -8ab + 2ka^2 - 6kb^2 - 4kab$$
$$y(a,b) = -2a^2 + 6b^2 - 4ab - 8kab$$
$$X(a,b) = -a^2 + 3b^2 - 14ab + 3ka^2 - 9kb^2 - 10kab \quad (10)$$
$$Y(a,b) = -3a^2 + 9b^2 - 10ab + ka^2 - 3kb^2 - 14kab$$
$$w(a,b) = a^2 + 3b^2$$

### 2.4 Properties

$$3X(a,a+1) - Y(a,a+1) + 16Pr_a(2+k) + k(7t_{4,a} + 30a + 15) \equiv 0$$
$$Y(a,2a^2-1) - 3w(a,2a^2-1) + 6t_{4,a}(1-2k) + k(-a + 12t_{4,a^2} + 3) + 2SO_a(5+7k) \equiv 0$$
$$x(a^2,a+1) - y(a^2,a+1) + 4t_{4,a}(-1+k) + 4P_a^5(7+6k) + 2k(6a+3) \equiv 6 \pmod{12}$$
$$y(a,2a^2+1) - 2w(a,2a^2+1) + 4t_{4,a} + 12OH_a(1+2k) \equiv 0$$
$$x(a,3a-1) - w(a,3a-1) + 2t_{4,a}(1+39k) + 4t_{5,a}(7+5k) + 3k(-18a+3) \equiv 0$$

### 2.5 Note

It is to be noted that in addition to (7), 1 may also represented as the product of complex conjugates as given below:

**Choice: 1**

$$1 = \frac{(-1+i\sqrt{3})(-1-i\sqrt{3})}{4}$$

**Choice: 2**

$$1 = \frac{(11+i4\sqrt{3})(11-i4\sqrt{3})}{169}$$

**Choice: 3**

$$1 = \frac{(-11+i4\sqrt{3})(-11-i4\sqrt{3})}{169}$$

**Choice: 4**

$$1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}$$

**Choice: 5**

$$1=\frac{\left(-1+i4\sqrt{3}\right)\left(-1-i4\sqrt{3}\right)}{49}$$

**Choice: 6**

$$1=\frac{\left(11+i5\sqrt{3}\right)\left(11-i5\sqrt{3}\right)}{196}$$

**Choice: 7**

$$1=\frac{\left(-11+i5\sqrt{3}\right)\left(-11-i5\sqrt{3}\right)}{196}$$

**Choice: 8**

$$1=\frac{\left(13+i3\sqrt{3}\right)\left(13-i3\sqrt{3}\right)}{196}$$

**Choice: 9**

$$1=\frac{\left(-13+i3\sqrt{3}\right)\left(-13-i3\sqrt{3}\right)}{196}$$

Employing the procedure presented in pattern:2, the corresponding integral solutions to (1) for the above choices of 1 are obtained as below;

**Solution for choice: 1**

$$x(a,b)=-2a^2+6b^2-4ab-8kab$$
$$y(a,b)=-2a^2+6b^2+4ab-2ka^2+6kb^2-4kab$$
$$X(a,b)=-4a^2+12b^2-4ab-ka^2+3kb^2-14kab$$
$$Y(a,b)=-4a^2+12b^2+4ab-3ka^2+9kb^2-10kab$$
$$\mathrm{w(a,b)}=a^2+3b^2$$

**Solution for choice: 2**

$$x(a,b)=14a^2-42b^2-48ab+30ka^2-90kb^2-48kab$$
$$y(a,b)=-16a^2+48b^2-132ab+14ka^2-42kb^2-48kab$$
$$X(a,b)=13a^2-39b^2-138ab+52ka^2-156kb^2-96kab$$
$$Y(a,b)=-17a^2+51b^2-222ab+36ka^2-108kb^2-96kab$$

**Solution for choice: 3**

$$x(a,b)=-30a^2+90b^2-4ab-14ka^2+42kb^2-92kab$$
$$y(a,b)=-16a^2+48b^2+88ab-30ka^2+90kb^2-4kab$$
$$X(a,b)=-53a^2+159b^2+38ab-36ka^2+108kb^2-140kab$$
$$Y(a,b)=-39a^2+117b^2+130ab-52ka^2+156kb^2-52kab$$
$$\mathrm{w(a,b)}=13a^2+39b^2$$

**Solution for choice: 4**

$$x(a,b)=-6a^2+18b^2-52ab-6ka^2-30kb^2-44kab$$
$$y(a,b)=-16a^2+48b^2-8ab+10ka^2+18kb^2-52kab$$
$$X(a,b)=-17a^2+51b^2-82ab-4ka^2-36kb^2-92kab$$
$$Y(a,b)=-27a^2+81b^2-38ab+12ka^2+12kb^2-100kab$$
$$\mathrm{w(a,b)}=7a^2+21b^2$$

**Solution for choice: 5**

$$x(a,b)=-10a^2+30b^2-44ab+6ka^2-18kb^2-52kab$$
$$y(a,b)=-16a^2+48b^2+8ab-10ka^2+30kb^2-44kab$$
$$X(a,b)=-23a^2+69b^2-62ab+4ka^2-12kb^2-100kab$$
$$Y(a,b)=-29a^2+87b^2-10ab-12ka^2+36kb^2-92kab$$
$$\mathrm{w(a,b)}=7a^2+21b^2$$

**Solution for choice: 6**

$$x(a,b)=12a^2-36b^2-104ab-12ka^2-96kb^2-16kab$$
$$y(a,b)=-20a^2+60b^2-88ab-32ka^2-36kb^2-104kab$$
$$X(a,b)=8a^2-24b^2-200ab-34ka^2-162kb^2-76kab$$
$$Y(a,b)=-24a^2+72b^2-184ab-54ka^2-102kb^2-164kab$$
$$\mathrm{w(a,b)}=14a^2+42b^2$$

**Solution for choice: 7**

$$x(a,b)=-32a^2+96b^2-16ab-12ka^2+36kb^2-104kab$$
$$y(a,b)=-20a^2+60b^2+88ab-12ka^2+96kb^2-16kab$$
$$X(a,b)=-58a^2+174b^2+20ab-34ka^2+102kb^2-164kab$$
$$Y(a,b)=-46a^2+138b^2+124ab-34ka^2+162kb^2-76kab$$
$$\mathrm{w(a,b)}=14a^2+42b^2$$

**Solution for choice: 8**

$$x(a,b)=20a^2-60b^2-88ab+32ka^2-96kb^2+16kab$$
$$y(a,b)=-12a^2+36b^2-104ab+20ka^2-60kb^2-88kab$$
$$X(a,b)=24a^2-72b^2-184ab+58ka^2-174kb^2-20kab$$
$$Y(a,b)=-8a^2+24b^2-200ab+46ka^2-138kb^2-124kab$$
$$\mathrm{w(a,b)}=14a^2+42b^2$$

**Solution for choice: 9**

$$x(a,b)=-32a^2+96b^2+16ab-20ka^2+60kb^2-88kab$$
$$y(a,b)=-12a^2+36b^2+104ab-32ka^2+96kb^2+16kab$$
$$X(a,b)=-54a^2+162b^2+76ab-46ka^2+138kb^2-124kab$$
$$Y(a,b)=-34a^2+102b^2+164ab-58ka^2+174kb^2-20kab$$
$$\mathrm{w(a,b)}=14a^2+42b^2$$

**2.6 Pattern: 3**

Rewrite (3) as

$$\mathrm{u}^2-(2k+1)^2w^2=3(w^2-v^2) \qquad (11)$$

(11) is written in the form of ratio as

$$\frac{u+(2k+1)w}{w+v}=\frac{3(w-v)}{u-(2k+1)w}=\frac{\alpha}{\beta},\beta>0$$

Which is equivalent to the system of double equations

$$\left.\begin{array}{r}\beta u-v\alpha+(2k\beta+\beta-\alpha)w=0\\ -\alpha u-3\beta v+(2k\alpha+\alpha+3\beta)w=0\end{array}\right\}\qquad(12)$$

Solving the above system (12) by applying the method of cross multiplication, we get

$$\begin{aligned}u&=-\alpha^2-2k\alpha^2+3\beta^2+6k\beta^2-6\alpha\beta\\ v&=\alpha^2-3\beta^2-4k\alpha\beta-2\alpha\beta\\ w&=-\alpha^2-3\beta^2\end{aligned}\qquad(13)$$

Using (13) and (2), we get the corresponding non-zero integer solutions of (1) to be

$$x(\alpha,\beta)=-2k\alpha^2+6k\beta^2-8\alpha\beta-4k\alpha\beta$$
$$y(\alpha,\beta)=-2\alpha-2k\alpha^2+6\beta^2+6k\beta^2-4\alpha\beta+4k\alpha\beta$$
$$X(\alpha,\beta)=-\alpha^2-4k\alpha^2+3\beta^2+12k\beta^2-14\alpha\beta-4k\alpha\beta$$
$$Y(\alpha,\beta)=-3\alpha^2-4k\alpha^2+9\beta^2+12k\beta^2-10\alpha\beta-4k\alpha\beta$$
$$w(\alpha,\beta)=-\alpha^2-3\beta^2$$

**2.7 Properties**

$$x(a,2a^2-1)-y(a,2a^2-1)-26t_{4,a}+24t_{4,a^2}+4SO_a(1+2k)+6\equiv 0$$

$$X(a,a)-Y(a,a)+8a^2(1+k)\equiv 0$$

$$X(a,a+1)+w(a,a+1)+2t_{4,a}(1-4k)+2Pr_a(7+2k)-24ka+12\equiv 0$$

$$Y(a^2,a+1)+3w(a^2,a+1)+t_{4,a}(3-8k)-12k(2a+1)+4P_a^5(5+2k)\equiv 0$$

$$y(a,2a-1)-2w(a,2a-1)+2t_{4,a}(2-11k)+4t_{6,a}(1-k)+6k(4a-1)\equiv 0$$

**2.8 Remark**

(11) may also be expressed in the form of ratios in three more different ways that are presented below:

**Way 1**

$$\frac{u+(2k+1)w}{3(w+v)}=\frac{(w-v)}{u-(2k+1)w}=\frac{\alpha}{\beta}$$

**Way 2**

$$\frac{u+(2k+1)w}{(w-v)}=\frac{3(w+v)}{u-(2k+1)w}=\frac{\alpha}{\beta}$$

**Way 3**

$$\frac{u+(2k+1)w}{3(w-v)}=\frac{(w+v)}{u-(2k+1)w}=\frac{\alpha}{\beta}$$

Solving each of the above system of equations by following the procedure as presented in
Pattern:3, the corresponding integer solutions to (1) are found to be as given below:

**Solution for way 1**

$$x(\alpha,\beta,k)=-8\alpha\beta-6k\alpha^2+2k\beta^2-4k\alpha\beta$$
$$y(\alpha,\beta,k)=-6\alpha^2+2\beta^2-4\alpha\beta-6k\alpha^2+2k\beta^2+4k\alpha\beta$$
$$X(\alpha,\beta,k)=-3\alpha^2+\beta^2-14\alpha\beta-12k\alpha^2+4k\beta^2-4k\alpha\beta$$
$$Y(\alpha,\beta,k)=-9\alpha^2+3\beta^2-10\alpha\beta-12k\alpha^2+4k\beta^2+4k\alpha\beta$$
$$w(\alpha,\beta)=-3\alpha^2-\beta^2$$

**Solution for way 2**

$$x(\alpha,\beta,k)=2\alpha^2-6\beta^2+4\alpha\beta+2k\alpha^2-6k\beta^2-4k\alpha\beta$$
$$y(\alpha,\beta,k)=2k\alpha^2-6k\beta^2+8\alpha\beta+4k\alpha\beta$$
$$X(\alpha,\beta,k)=3\alpha^2-9\beta^2+10\alpha\beta+4k\alpha^2-12k\beta^2-4k\alpha\beta$$
$$Y(\alpha,\beta,k)=\alpha^2-3\beta^2+14\alpha\beta+4k\alpha^2-12k\beta^2+4k\alpha\beta$$
$$w(\alpha,\beta)=\alpha^2+3\beta^2$$

**Solution for way 3**

$$x(\alpha,\beta,k)=6\alpha^2+6k\alpha^2+2k\beta^2-2\alpha\beta-4k\alpha\beta$$
$$y(\alpha,\beta,k)=2\beta^2+6k\alpha^2+2k\beta^2+2\alpha\beta+4k\alpha\beta$$
$$X(\alpha,\beta,k)=9\alpha^2+\beta^2+12k\alpha^2+4k\beta^2-2\alpha\beta-4k\alpha\beta$$
$$Y(\alpha,\beta,k)=3\alpha^2+3\beta^2+12k\alpha^2+4k\beta^2+2\alpha\beta+4k\alpha\beta$$
$$w(\alpha,\beta)=3\alpha^2+\beta^2$$

**3. Conclusion**

In this paper, we have presented different choices of integer solutions to the homogeneous biquadratic equation with five unknowns $(x-y)(x^3+y^3)=(2k^2+2k+2)(X^2-Y^2)w^2$. It is worth mentioning here that in (2), the linear transformations for X and Y may also be considered as I) X=uv+2, Y=uv-2 and II) X=2uv+1, Y=2uv-1. Employing the above two forms of transformations for X,Y different values for X and Y are obtained. To conclude, as biquadratic equations are rich in variety, one may consider other forms of biquadratic equations and search for corresponding properties.

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