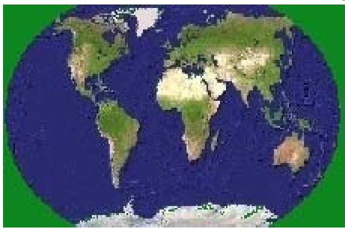

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A study on Henkel transform and its relation to the Fourier transform

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Abstract
In mathematics, the Hankel transform expresses any given function f(r) as the weighted sum of an infinite number of Bessel functions of the first kind $J_\nu(kr)$. The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor k along the r-axis. The necessary coefficient F_ν of each Bessel function in the sum, as a function of the scaling factor k constitutes the transformed function.

Keywords: Hankel Transform, Fourier Transform.

Introduction

The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel. It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval.
The Hankel transform of order ν of a function *f*(*r*) is given by:

$$F_\nu(k) = \int_0^\infty f(r) J_\nu(kr)\, r\, \mathrm{d}r$$

Where

J_ν is the Bessel function of the first kind of order ν with $\nu \geq -\frac{1}{2}$

The inverse Hankel transform of $F_\nu(k)$ is defined as:

$$f(r) = \int_0^\infty F_\nu(k) J_\nu(kr) k\, \mathrm{d}k$$

Which can be readily verified using the orthogonality relationship described below.
Inverting a Hankel transform of a function *f*(*r*) is valid at every point at which *f*(*r*) is continuous provided that the function is defined in (0, ∞), is piecewise continuous and of bounded variation in every finite subinterval in (0, ∞), and

$$\int_0^\infty |f(r)|\, r^{\frac{1}{2}}\, \mathrm{d}r < \infty.$$

However, like the Fourier Transform, the domain can be extended by a density argument to include some functions whose above integral is not finite, for example

$$f(r) = (1 + r)^{-3/2}.$$

An alternative definition says that the Hankel transform of *g*(*r*) is:

$$h_\nu(k) = \int_0^\infty g(r) J_\nu(kr) \sqrt{kr}\, \mathrm{d}r$$

The two definitions are related:

$$\text{If } g(r) = f(r)\sqrt{r} \text{ then } h_\nu(k) = F_\nu(k)\sqrt{k}.$$

This means that, as with the previous definition, the Hankel transform defined this way is also its own inverse:

$$g(r) = \int_0^\infty h_\nu(k) J_\nu(kr)\sqrt{kr}\,\mathrm{d}k$$

The obvious domain now has the condition

$$\int_0^\infty |g(r)|\,\mathrm{d}r < \infty$$

But this can be extended.
According to the reference given above, we can take the integral as the limit as the upper limit goes to infinity (an improper integral rather than a Lebesgue integral) and in this way the Hankel transform and its inverse work for all functions in $L^2(0, \infty)$.
The Bessel functions form an orthogonal basis with respect to the weighting factor r:

$$\int_0^\infty J_\nu(kr) J_\nu(k'r) r\,\mathrm{d}r = \frac{\delta(k-k')}{k}, \qquad k, k' > 0.$$

The Plancherel Theorem and Parseval's Theorem
If $f(r)$ and $g(r)$ are such that their Hankel transforms $F_\nu(k)$ and $G_\nu(k)$ are well defined, then the Plancherel theorem states

$$\int_0^\infty f(r)g(r)r\,\mathrm{d}r = \int_0^\infty F_\nu(k)G_\nu(k)k\,\mathrm{d}k.$$

Parseval's theorem, which states:

$$\int_0^\infty |f(r)|^2 r\,\mathrm{d}r = \int_0^\infty |F_\nu(k)|^2 k\,\mathrm{d}k,$$

Is a special case of the Plancherel theorem? These theorems can be proven using the orthogonality property.

Relation to the Fourier Transform (Circularly Symmetric Case)
The Hankel transform of order zero is essentially the 2-dimensional Fourier transform of a circularly symmetric function.
Consider a 2-dimensional function f ($\mathbf{r}$) of the radius vector $\mathbf{r}$. Its Fourier transform is:

$$F(\mathbf{k}) = \iint f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r}.$$

With no loss of generality, we can pick a polar coordinate system (r, θ) such that the $\mathbf{k}$ vector lies on the $\theta = 0$ axis (in K-space). The Fourier transform is now written in these polar coordinates as:

$$F(\mathbf{k}) = \int_{r=0}^\infty \int_{\theta=0}^{2\pi} f(r,\theta)e^{ikr\cos(\theta)}\,r\,\mathrm{d}\theta\,\mathrm{d}r$$

Where θ is the angle between the $\mathbf{k}$ and $\mathbf{r}$ vectors. If the function f happens to be circularly symmetric, it will have no dependence on the angular variable θ and may be written $f(r)$. The integration over θ may be carried out, and the Fourier transform is now written:

$$F(\mathbf{k}) = F(k) = 2\pi\int_0^\infty f(r)J_0(kr)r\,\mathrm{d}r$$

Which is just 2π times the zero-order Hankel transform of f (r). For the reverse transform,

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2}\iint F(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{k} = \frac{1}{2\pi}\int_0^\infty F(k)J_0(kr)k\,\mathrm{d}k$$

So f (r) is $1/2\pi$ times the zero-order Hankel transform of F (k).

Relation to the Fourier Transform (Radially Symmetric Case in *N*-Dimensions)

For an n-dimensional Fourier transform,

$$F(\mathbf{k}) = \int f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^n\mathbf{r}$$

if the function f is radially symmetric, then:

$$k^{n/2-1}F(k) = (2\pi)^{n/2}\int_0^\infty r^{n/2-1}f(r)J_{n/2-1}(kr)r\,dr$$

Relation to the Fourier Transform (General Case)
To generalize: If f can be expanded in a multipole series,

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} f_m(r)e^{im\theta},$$

And if θ_k is the angle between the direction of $\mathbf{k}$ and the $\theta = 0$ axis,

$$\begin{aligned}
F(\mathbf{k}) &= \int_0^\infty r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\, f(r,\theta)e^{ikr\cos(\theta-\theta_k)}\\
&= \sum_m \int_0^\infty r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\, f_m(r)e^{im\theta}e^{ikr\cos(\theta-\theta_k)}\\
&= \sum_m e^{im\theta_k}\int_0^\infty r\,\mathrm{d}r\, f_m(r)\int_0^{2\pi}\mathrm{d}\varphi\, e^{im\varphi}e^{ikr\cos\varphi} \qquad \varphi = \theta-\theta_k\\
&= \sum_m e^{im\theta_k}\int_0^\infty r\,\mathrm{d}r\, f_m(r)2\pi i^m J_m(kr)\\
&= 2\pi\sum_m i^m e^{im\theta_k}\int_0^\infty f_m(r)J_m(kr)r\,\mathrm{d}r.\\
&= 2\pi\sum_m i^m e^{im\theta_k}F_m(k)
\end{aligned}$$

Where $F_m(k)$ is the m-th order Hankel transform of $f_m(r)$.

Functions inside a Limited Radius
Additionally, if f_m is sufficiently smooth near the origin and zero outside a radius R, it may be expanded into a Chebyshev series,

$$f_m(r) = r^m \sum_{t \geq 0} f_{mt} \left(1 - \left(\tfrac{r}{R}\right)^2\right)^t, \qquad 0 \leq r \leq R.$$

So that

$$F(\mathbf{k}) = 2\pi \sum_m i^m e^{im\theta_k} \sum_t f_{mt} \int_0^R r^m \left(1 - \left(\tfrac{r}{R}\right)^2\right)^t J_m(kr) r \, dr \qquad (*)$$

$$= 2\pi \sum_m i^m e^{im\theta_k} R^{m+2} \sum_t f_{mt} \int_0^1 x^m (1 - x^2)^t J_m(kxR) x \, dx \quad x = \tfrac{r}{R}$$

$$= 2\pi \sum_m i^m e^{im\theta_k} R^{m+2} \sum_t f_{mt} \frac{t! 2^t}{(kR)^{1+t}} J_{m+t+1}(kR).$$

The above can be viewed as a more general case that is not as constrained as the previous case in the previous section. The numerically important aspect is that the expansion coefficients f_{mt} are accessible with Discrete Fourier transform techniques. Insertion into the previous formula yields. This is one flavor of fast Hankel transform techniques.

Relation to the Fourier and Abel Transforms

The Hankel transform is one member of the FHA cycle of integral operators. In two dimensions, if we define A as the Abel transform operator, F as the Fourier transform operator and H as the zeroth order Hankel transform operator, then the special case of the projection-slice theorem for circularly symmetric functions states that:

$$FA = H.$$

In other words, applying the Abel transform to a 1-dimensional function and then applying the Fourier transform to that result is the same as applying the Hankel transform to that function. This concept can be extended to higher dimensions.

The *k-binomial transform* W *of a sequence* A is the sequence $W(A,k) = \{w_n\}$, where w_n is given by

$$w_n = \begin{cases} \sum_{i=0}^n \binom{n}{i} k^n a_i = k^n \sum_{i=0}^n \binom{n}{i} a_i, & \text{if } k \neq 0 \text{ or } n \neq 0; \\ a_0, & \text{if } k = 0, n = 0. \end{cases}$$

The *rising k-binomial transform* R *of a sequence* A is the sequence $R(A,k) = \{r_n\}$, where r_n is given by

$$r_n = \begin{cases} \sum_{i=0}^n \binom{n}{i} k^i a_i, & \text{if } k \neq 0; \\ a_0, & \text{if } k = 0. \end{cases}$$

The *falling k-binomial transform* F *of a sequence* A is the sequence $F(A,k) = \{f_n\}$, where f_n is given by

$$f_n = \begin{cases} \sum_{i=0}^n \binom{n}{i} k^{n-i} a_i, & \text{if } k \neq 0; \\ a_n, & \text{if } k = 0. \end{cases}$$

The case k = 0 must be dealt with separately because 0 0 would occur in the formulas otherwise. Our definitions effectively take 0 0 to be 1. These turn out to be "good" definitions, in the sense that all the results discussed subsequently hold under our definitions for the k = 0 case. When k = 0, the k-binomial transform of A is the sequence {a0, 0, 0, 0,...}, the rising k-binomial transform of A is {a0, a0, a0,...}, and the falling k-binomial transform is the identity transform.

The k-binomial transform when k = 1/2 is of special interest; this is the binomial mean transform, defined in the OEIS [11], sequence A075271. When k is a positive integer, these variations of the binomial transform all have combinatorial interpretations similar to that of the binomial transform, although, unlike the binomial transform, they have a two-dimensional component.

If an represents the number of arrangements of n labeled objects with some property P, then w_n represents the number of ways of dividing n objects such that

- In one dimension, the n objects are divided into two groups so that the first group has property P.
- In a second dimension, the n objects are divided into k labeled groups.

The interpretation of the second dimension could be something as simple as a coloring of each object from a choice of k colors, independent of the division of the objects in the first dimension. For example, if the input sequence is the derangement numbers, w_n is the number of ways of dividing n labeled objects into two groups such that the objects in the first group are deranged and each of the n objects has been colored one of k colors, independently of the initial division into two groups.

With this interpretation of w_n in mind, r_n represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and each object in the first group is further placed into one of k labeled groups (e.g., colored using one of k colors). Similarly, fn represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and the objects in the second group are further placed into k labeled groups (e.g., colored using one of k colors).

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