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Observations on the Biquadratic Equation with five unknowns *(X-Y) (x³+Y³) = 39(W²-Z²) P²*

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Abstract
The biquadratic Diophantine equation with five unknowns $(x-y)(x^3+y^3)=39(w^2-z^2)p^2$ is analysed for its infinitely many non-zero distinct integral solutions. A few interesting properties among the values of x, y, z, w, p and special numbers namely, polygonal, Centered pyramidal, Jacobsthal numbers, and Four dimensional figurate numbers are presented.

Keywords: biquadratic equation with five unknowns, integral solutions
MSC 2010 Mathematics Subject Classification: 11D25

Notations Used:

- $t_{m,n} = n\left(1+\frac{(n-1)(m-2)}{2}\right)$ - Polygonal number of rank n with size m
- $CP_n^m = m\left[\frac{(n-1)n(n+1)}{6}\right]+n$ - Centered Pyramidal number of rank n with size m
- $Pr_n = n(n+1)$ -Pronic number of rank n
- $J_n = \frac{1}{3}(2^n-(-1)^n)$ - Jacobsthal number of rank n
- $F_{4,3}^n = \frac{1}{24}(n^4+6n^3+11n^2+6n)$ - Four dimensional figurate number of rank n whose generating polygon is a triangle
- $F_{4,4}^n = \frac{1}{12}(n^4+4n^3+5n^2+2n)$ - Four dimensional figurate number of rank n whose generating polygon is a square
- $F_{4,5}^n = \frac{1}{24}(3n^4+10n^3+9n^2+2n)$ - Four dimensional figurate number of rank n whose generating polygon is a pentagon

$F_{4,6}^n = \frac{1}{6}(n^4+3n^3+2n^2)$ - Four dimensional figurate number of rank n whose generating polygon is a hexagon

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular Biquadratic equations, homogeneous or non-homogeneous, have aroused the interest of numerous mathematicians since antiquity [1-3]. For illustration, one may refer [4-15] for Biquadratic equations with three and four unknowns. This paper concerns with the problem of determining non-trivial integral solutions of the homogeneous Biquadratic equation with five unknowns given by $(x-y)(x^3+y^3)=39(w^2-z^2)p^2$. A few relations between the solutions and the special numbers are presented.

2. Method of Analysis

The Diophantine equation representing the Biquadratic equation with five unknowns under consideration is $(x-y)(x^3+y^3)=39(w^2-z^2)p^2$ (1)

Introducing the transformations

$$x=u+v, y=u-v, w=uv+1, z=uv-1 \quad (2)$$

in (1), we get $u^2 + 3v^2 = 39p^2$ (3)

Now, we solve (3) through different methods and thus obtain different patterns of solutions to (1).

Pattern I

Equation (3) can be written as

$$\frac{6p+u}{v+p} = \frac{3(v-p)}{6p-u} = \frac{A}{B} \quad where\ B \neq 0 \tag{4}$$

This equation is equivalent to the following two equations

$$uB - Av + p(6B - A) = 0, \; uA + 3vB - p(3B + 6A) = 0 \tag{5}$$

By the method of cross multiplication we get the integral solutions of (5) to be

$$u = 6A^2 - 18B^2 + 6AB$$
$$v = -A^2 + 3B^2 + 12AB$$
$$p = A^2 + 3B^2 \tag{5a}$$

In view of (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$x = x(A,B) = 5A^2 + 18AB - 15B^2$$
$$y = y(A,B) = 7A^2 - 6AB - 21B^2$$
$$w = w(A,B) = (6A^2 - 18B^2 + 6AB)(-A^2 + 3B^2 + 12AB) + 1$$
$$z = z(A,B) = (6A^2 - 18B^2 + 6AB)(-A^2 + 3B^2 + 12AB) - 1$$

along with (5a).

A few interesting properties observed are as follows:

1. $3y(B+1,B) - 6x(B+1,B) + 9p(B+1,B) + 126Pr_B$ is a Nasty number
2. $16y(A,1) - w(A,1) - 72F_{4,4}^{A} + 18CP_A^{30} + 4t_{15,A} \equiv -3 \pmod{4}$
3. $x(A,1) + z(A,1) + 5p(A,1) + 144F_{4,3}^{A} - 36CP_A^{17} - 23t_{18,A} \equiv -55 \pmod{83}$
4. $147\left(w(A,A) + z(A,A) + 1008F_{4,6}^{A} - 336t_{4,A}\right)$ is a cubical integer
5. $p(2^n,1) = 3J_{2n} + 4$

Pattern II

Equation (3) can also be written as

$$\frac{6p+u}{3(v+p)} = \frac{(v-p)}{6p-u} = \frac{A}{B} \quad where\ B \neq 0$$

Proceeding as in the above pattern I, the corresponding non-zero distinct integral solutions of (1) are found to be

$$x = x(A,B) = 15A^2 - 5B^2 + 18AB$$
$$y = y(A,B) = 21A^2 - 7B^2 - 6AB$$
$$w = w(A,B) = (18A^2 - 6B^2 + 6AB)(-3A^2 + B^2 + 12AB) + 1$$
$$z = z(A,B) = (18A^2 - 6B^2 + 6AB)(-3A^2 + B^2 + 12AB) - 1$$
$$p = p(A,B) = B^2 + 3A^2$$

Properties:

1. $39[7x(A,A+1) - 5y(A,A+1) - 117Pr_A - 39A]$ is a perfect square
2. $10x(A,A^2) - 4p(A,A^2) - w(a,1) + 6CP_A^{18} - 6t_{12,A} \equiv 5 \pmod{78}$
3. $x(B+1,B) + 3y(B+1,B) - 8t_{15,B} \equiv 78 \pmod{200}$
4. $z(A,1) - 5y(A,1) + 324F_{4,6}^{A} - 108CP_A^{20} - 3Pr_A \equiv 28 \pmod{321}$
5. $p(2^n,1) = 9J_{2n} + 4$

Pattern III

Equation (3) can be written as $39p^2 - u^2 = 3v^2$ (6)

Assume $v = 39A^2 - B^2$ (7)

and write 3 as $3 = (\sqrt{39}+6)(\sqrt{39}-6)$ (8)

Using (7) and (8) in (6), it is written in the factorizable form as

$$(\sqrt{39}p + u)(\sqrt{39}p - u) = (\sqrt{39}+6)(\sqrt{39}-6)(\sqrt{39}A + B)^2(\sqrt{39}A - B)^2$$

which is equivalent to the system of equations

$$(\sqrt{39}p + u) = (\sqrt{39}+6)(\sqrt{39}A + B)^2$$
$$(\sqrt{39}p - u) = (\sqrt{39}-6)(\sqrt{39}A - B)^2$$

Solving either of the above two equations, we have

$$u = 234A^2 + 6B^2 + 78AB$$
$$v = 39A^2 - B^2$$
$$p = 39A^2 + B^2 + 12AB \tag{8a}$$

In view of (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$x = x(A,B) = 273A^2 + 5B^2 + 78AB$$
$$y = y(A,B) = 195A^2 + 7B^2 + 78AB$$
$$w = w(A,B) = (234A^2 + 6B^2 + 78AB)(39A^2 - B^2) + 1$$
$$z = z(A,B) = (234A^2 + 6B^2 + 78AB)(39A^2 - B^2) - 1$$

along with (8a).

Properties:

1. $7x(A,A+1) - 5y(A,A+1) - 1092Pr_A \equiv 0 \pmod{936}$

2. $12[y(B+1,B) - 5p(B+1,B) - 18Pr_B]$ is a Nasty number

3. $w(1,B) + 72F_{4,4}^{B} - 6CP_B^{24} + t_{14,B} \equiv 31 \pmod{3032}$

4. $2x(1,2^n) - 13p(1,2^n) = 36 - 9J_{2n}$

Pattern IV

Assume $p = p(a,b) = a^2 + 3b^2$ (9)

where a and b are non-zero distinct integers.

Write $39 = (6 + i\sqrt{3})(6 - i\sqrt{3})$ (10)

Using (9) and (10) in (3), and applying the method of factorization, it is written as the system of double equations as

$$u + i\sqrt{3}v = (6 + i\sqrt{3})(a + i\sqrt{3}b)^2$$
$$u - i\sqrt{3}v = (6 - i\sqrt{3})(a - i\sqrt{3}b)^2$$

Equating the real and imaginary parts of either of the above equations, we have

$$u = u(a,b) = 6a^2 - 18b^2 - 6ab$$
$$v = v(a,b) = a^2 - 3b^2 + 12ab$$

In view of (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$x = x(a,b) = 7a^2 - 21b^2 + 6ab$$
$$y = y(a,b) = 5a^2 - 15b^2 - 18ab$$
$$w = w(a,b) = (6a^2 - 18b^2 - 6ab)(a^2 - 3b^2 + 12ab) + 1$$
$$z = z(a,b) = (6a^2 - 18b^2 - 6ab)(a^2 - 3b^2 + 12ab) - 1$$

along with (9).

Properties:

1. $x(a,a+1)-y(a,a+1)+2p(a,a+1)-24Pr_a$ is a perfect square
2. $2y(b+1,b)+4p(b+1,b)-2x(b+1,b)+48Pr_b$ is a Nasty number
3. $w(a,1)+5x(a,1)-36F_{4,6}^{a}-18CP_a^{16}+34t_{7,a}\equiv -50(mod189)$
4. $z(a,1)+y(a,1)-48F_{4,5}^{a}-12CP_a^{23}+22t_{13,a}\equiv 38(mod285)$
5. $w(a+2,a)-z(a+2,a)+p(a+2,a)-4Pr_a=6$

Note :

It is seen that 39 may also be represented in the following ways:

(i). $39=\frac{1}{4}(9+i5\sqrt{3})(9-i5\sqrt{3})$

(ii). $39=\frac{1}{4}(3+i7\sqrt{3})(3-i7\sqrt{3})$

(iii). $39=\frac{1}{49}(6+i25\sqrt{3})(6-i25\sqrt{3})$ (11)

(iv). $39=\frac{1}{49}(18+i23\sqrt{3})(18-i23\sqrt{3})$

For each of the above representation, the corresponding solutions to (1) are as follows:

Solution for (i):

$$x=x(a,b)=28a^2-84b^2-24ab$$
$$y=y(a,b)=8a^2-24b^2-96ab$$
$$w=w(a,b)=4[9(a^2-3b^2)-30ab][5(a^2-3b^2)+18ab]+1$$
$$z=z(a,b)=4[9(a^2-3b^2)-30ab][5(a^2-3b^2)+18ab]-1$$
$$p=p(a,b)=4(a^2+3b^2)$$

Solution for (ii):

$$x=x(a,b)=20a^2-60b^2-72ab$$
$$y=y(a,b)=-8a^2+24b^2-96ab$$
$$w=w(a,b)=4[3(a^2-3b^2)-42ab][7(a^2-3b^2)+6ab]+1$$
$$z=z(a,b)=4[3(a^2-3b^2)-42ab][7(a^2-3b^2)+6ab]-1$$
$$p=p(a,b)=4a^2+12b^2$$

Solution for (iii):

$$x=x(a,b)=217a^2-651b^2-966ab$$
$$y=y(a,b)=-133a^2+399b^2-1134ab$$
$$w=w(a,b)=(42a^2-126b^2-1050ab)(175a^2-525b^2+84ab)+1$$
$$z=z(a,b)=(42a^2-126b^2-1050ab)(175a^2-525b^2+84ab)-1$$
$$p=p(a,b)=49a^2+147b^2$$

Solution for (iv):

$$x=x(a,b)=287a^2-861b^2-714ab$$
$$y=y(a,b)=-35a^2+105b^2-1218ab$$
$$w=w(a,b)=(126a^2-378b^2-966ab)(161a^2-483b^2+252ab)+1$$
$$z=z(a,b)=(126a^2-378b^2-966ab)(161a^2-483b^2+252ab)-1$$
$$p=p(a,b)=49a^2+147b^2$$

Pattern V

Equation (3) can be also written as

$$u^2+3v^2==39p^2=39p^2*1$$

Write 1 as $1=\frac{1}{4}(1+i\sqrt{3})(1-i\sqrt{3})$ (12)

Applying a similar analysis presented in above pattern IV and performing a few calculations, the corresponding non-zero distinct integral solutions to (1) are given by

$$x=x(a,b)=133a^2-399b^2-1134ab$$
$$y=y(a,b)=-217a^2+651b^2-966ab$$
$$w=w(a,b)=(-42a^2+126b^2-1050ab)(175a^2-525b^2-84ab)+1$$
$$z=z(a,b)=(-42a^2+126b^2-1050ab)(175a^2-525b^2+84ab)-1$$
$$p=p(a,b)=49a^2+147b^2$$

Properties:

1. $x(a,1)+y(a,1)+p(a,1)+5t_{16,a}\equiv 399(mod2130)$
2. $y(1,b)+x(1,b)-p(1,b)-14t_{17,b}\equiv -133(mod2009)$
3. $z(a,1)+88200F_{4,4}^{a}+33516CP_a^{27}-33810t_{12,a}\equiv -66151(mod573300)$
4. $p(2^n,1)=147J_{2n}+196$

Note:

It is seen that 1 can also be represented in the following ways:

(i). $1=\frac{1}{4}(1+i\sqrt{3})(1-i\sqrt{3})$

(ii). $1=\frac{1}{676}(1+i15\sqrt{3})(1-i15\sqrt{3})$

(13)

(iii). $1=\frac{1}{169}(11+i4\sqrt{3})(11-i4\sqrt{3})$

By considering combinations between 39 and 1 given in (10), (11), (12) and (13) and performing analysis similar to that presented above, one may get other different choices of solutions to (1).

3. Conclusion

In this paper, we have obtained different patterns of non-zero distinct integer solutions to the Biquadratic equation with five unknowns given by $(x-y)(x^3+y^3)=39(w^2-z^2)p^2$. As biquadratic equations are rich in variety, one may search for integer solutions to biquadratic equations with variables ≥ 5 and determine their corresponding properties involving polygonal, pyramidal and other special number patterns.

4. References:

1. Carmichael, R.D., The theory of numbers and Diophantine Analysis, Dover Publications, New York, 1959
2. Dickson L.E., History of Theory of Numbers , vol-2,Chelsea Publishing Company, New York, 1952
3. Mordel L. J., Diophantine Equations , Academic Press, New York, 1969
4. Gopalan M. A, Pandichelvi V, On the solutions of the Biquadratic equation $(x^2-y^2)^2=(z^2-1)^2+w^4$, International Conference on Mathematical Methods and Computation at Jamal Mohamed College, 2009, 24-25, (July)

5. Gopalan M. A, Sangeetha G, Integral solutions of non-homogeneous Quartic equation $x^4 - y^4 = (2\alpha^2 + 2\alpha + 1)(z^2 - w^2)$, Imact J.Sci., Tech., 2010, 4(3) (July-Sep), 15-21
6. Gopalan M. A, Sangeetha G, Integral solutions of Ternary Quartic equation $x^2 - y^2 + 2xy = z^4$, Antartica J. Math., 2010, 7(1),95-101
7. Gopalan M. A, Sanmuganandham P, On the Biquadratic equation $x^4 + y^4 + z^4 = 2w^4$, Imact J.Sci., Tech., 2010, 4(4), 111-115
8. Gopalan M. A, Padma R, Integral solutions of non-homogeneous Quartic equation $x^4 - y^4 = (z^2 - w^2)$, Antartica J. Math., 2010, 7(4), 371-377
9. Gopalan M. A, Manju Somanath, Sangeetha G, Integral solutions of non-homogeneous Quartic equation $x^4 - y^4 = (k^2 + 1)(z^2 - w^2)$, Archimedes J. Math, 2011,1(1), 51-57
10. Gopalan M. A, Sanmuganandham P, On the Biquadratic equation $x^4 + y^4 + (x + y)z^3 = 2(k^2 + 3)^{2n}w^4$, Bessel J. Math., 2012, 2(2), 87-91
11. Gopalan M. A, Vidhyalakshmi S, Lakshmi K, On the Biquadratic equation with four unknowns $x^2 + xy + y^2 = (z^2 + zw + w^2)^2$, IJPAMS, 2012, 5(1), 73-77
12. Gopalan M. A, Sumathi G, Vidhyalakshmi S, Lattice points of non—homogeneous Biquadratic equation with four unknowns $x^4 - y^4 = 3z(x^3 + y^3) + w$, International Journal of Latest Research in Science and Technology, 2013, 2(1), 502-504
13. Gopalan M. A, Vidhyalakshmi S, Kavitha A, Integral solutions to the Biquadratic equation with four unknowns $(x + y + z + w)^2 = xyzw + 1$, IOSR-JM, 2013, 7(4), 11-13
14. Gopalan M. A, Sivakami B, Integral solutions of quartic equation with four unknowns $x^3 + y^3 + z^3 = 3xyz + 2(x + y)w^3$, Antartica J. Math., 2013,10(2), 151-159
15. Meena K, Vidhyalakshmi S, Gopalan M.A, Aarthy Thangam S, On the Biquadratic equation with four unknowns $x^3 + y^3 = 39zw^3$, IJOER-Online, 2014, 2(1), 57-60