International Journal of Multidisciplinary Research and Development

IJMRD 2015; 2(3): 919-920
www.allsubjectjournal.com
Received: 26-01-2015
Accepted: 28-02-2015
e-ISSN: 2349-4182
p-ISSN: 2349-5979
Impact Factor: 3.762

Mukesh Yadav
Asstt. Prof. of Mathematics
Govt. College Kanina
(M/Garh), Haryana, India

Correspondence
Mr. Mukesh Yadav
Asstt. Prof. of Mathematics
Govt. College Kanina
(M/Garh), Haryana, India

A brief introduction of Riemann integral

Mukesh Yadav

Abstract
In the branch of mathematics known as real analysis, the Riemann integral, created by Bernhard Riemann, was the first rigorous definition of the integral of a function on an interval. It was presented to the faculty at the University of Gottingen in 1854, but not published in a journal until 1868. For many functions and practical applications, the Riemann integral can be evaluated by the fundamental theorem of calculus or approximated by numerical integration. The Riemann integral is unsuitable for many theoretical purposes. Some of the technical deficiencies in Riemann integration can be remedied with the Riemann–Stieltjes integral, and most disappear with the Lebesgue integral. Historically, the concept of Integration came into existence as a means of evaluating areas under a curve, i.e., in compliance with a geometrical need. The first rigorous approach was therefore, quite naturally based on intuitive ideas of a sum and in effect as the limit of sum, now-a-days known as Riemann sum. But when the limitations of this approach were exposed through different situations, a rigorous arithmetic approach was contemplated by G.F.B. Riemann (1826–1866) with remarkable success. This approach was known as Riemann theory of integration which plays a fundamental role in analysis. Although it can be assumed that readers are familiar with the concepts of bounded and topology of real numbers, even then in view of the importance and utility of these topics for the study of Riemann integration, we give some fundamental definitions and proof of main theorem.

Keywords: riemann integral, stieltjes integral, lebesgue integral

1. Introduction
Partition
Let I = $[a, b]$ be a finite closed interval. Then by a partition P of I, we mean a finite ordered set $\{x_0, x_1, ..., x_n\}$ of points of $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < ..... < x_{n-1} < x_n = b$$

The partition P consists of $n + 1$ points and does a partitioning of $[a, b]$ into n sub-intervals (or segments)

$$I = [a, b] = \cup_{r=1}^{n} [x_{r-1}, x_r] \text{ Where } I_r = [x_{r-1}, x_r]$$

If length of the rth sub-interval $I_r = [x_{r-1}, x_r]$
is denoted by δ_r, then $\delta_r = x_r - x_{r-1}$, $(r = 1, 2, ..., n)$.

The greatest of the lengths of the sub-intervals of a partition P is called the **norm** or **mesh** of the partition P and is denoted by $||P||$ or μ (P).Thus, $||P|| = \max. \{(x_r - x_{r-1}) : 1 \le r \le n\}$

$$= \max. \{\delta_r : 1 \le r \le n\}$$

Let P be a partition of $[a, b]$, then any partition P′ is called a refinement of P if $P' \supseteq P$, *i.e.*, every point of P is a point of P′ and may have some other points of $[a, b]$ in it.Thus if
$P = \{a = x_0, x_1, ..., x_{r-1}, x_r, ..., x_n = b\}$ be a partition, then the partition $P' = \{a = x_0, x_1, ..., x_{r-1}, \xi_r, x_r, ..., x_n = b\}$ is a refinement of P having one point ξ_r, more than P.

If P_1 and P_2 are two partitions of $[a, b]$ and P_1, P_2 are both subsets of $P_1 \cup P_2$, then $P^* = P_1 \cup P_2$ is called a common refinement of P_1 and P_2.

Riemann Integral (Integration of Bounded Functions on R)

It was George Friedrich Bernhard Riemann (1826-1866), a German mathematician who gave the purely arithmetic treatment to the integration of a real valued bounded function de-fined on a finite closed interval

Let *f* be a real valued bounded function defined on a finite closed interval $[a, b]$ and suppose $P = \{a = x_0 < x_1 < ..... < x_{r-1} < x_r < .... x_n = b\}$ be any partition of $[a, b]$. Evidently, the function *f* is bounded on each sub-interval $[x_{r-1}, x_r]$, $(r = 1, 2, ..., n)$.

Let $m_r = \text{g.l.b. of } f \text{ in} [x_{r-1}, x_r]$

$$M_r = \text{l.u.b. of } f \text{ in} [x_{r-1}, x_r]$$

$$m = \text{g.l.b. of } f \text{ in } [a, b]$$

$$M = \text{l.u.b. of } f \text{ in} [a, b]$$

Clearly, $m \le m_r \le f(x) \le M_r \le M$, (r = 1, 2, ..., n)

Then, the sums

$$\sum m_r \delta_r = m_1\delta_1 + m_2\delta_2 + ... + m_n\delta_n$$

and $\sum M_r \delta_r = M_1\delta_1 + M_2\delta_2 + ... + M_n\delta_n$

are respectively called the lower and upper (Darboux) sums of *f* corresponding to the partition P and are denoted by L (*f*, P) and U (*f*, P) respectively.

Clearly, these sums depend upon the function *f* and the partition P and do exist for every bounded function. But the interval [*a*, *b*] can be divided into sub-intervals in infinitely many ways and for each partition of [*a*, *b*], we have lower and upper sums of *f*. considering all partitions of [*a*, *b*], we get a set of lower sums, *i.e.*, {L (*f*, P)} and a set of upper sums *i.e.*, {U (*f*, P)} of *f*. These sets are bounded and hence each has the g.l.b. and l.u.b.

The l.u.b. of the set of lower sums and the g.l.b. of the set of upper sums is called the ***lower*** and ***upper integral*** of *f* on [*a*, *b*] respectively and are denoted by $\int_{\bar{a}}^{b} f\, dx$ and $\int_{a}^{\bar{b}} f\, dx$

Thus $\int_{\bar{a}}^{b} f\, dx = l.u.b.\{L(f, P)\}$

and $\int_{a}^{\bar{b}} f\, dx = g.l.b.\{U(f, P)\}$

When these two integrals are equal, then we say that *f* is Riemann integrable or just integrable on [*a*, *b*] and the common value is called the Riemann integral of *f* on [*a*, *b*] and we write

$$\int_{a}^{b} f\, dx = \int_{\bar{a}}^{b} f\, dx = \int_{a}^{\bar{b}} f\, dx$$

and f is R-integrable.

References

1. Shilov GE, Gurevich BL. 'Integral, Measure, and Derivative: A Unified Approach', Richard A. Silverman, trans. Dover Publications, 1978. ISBN 0-486-63519-8.
2. Brown AB. A Proof of the Lebesgue Condition for Riemann Integrability. *The American Mathematical Monthly*. 1936; 43(7):396-398.
3. Basic real analysis, by Houshang H. Sohrab, section 7.3, Sets of Measure Zero and Lebesgue's Integrability Condition.
4. Ferreirós J. The Riemann Integral. In Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics. Basel, Switzerland: Birkhäuser. 1999, 150-153.
5. Jeffreys H, Jeffreys BS. Integration: Riemann, Stieltjes. In Methods of Mathematical Physics, 3rd ed. Cambridge, England: Cambridge University Press. 26-36.
6. Kestelman, H. Riemann Integration. Ch. 2 in Modern Theories of Integration, 2nd rev. Ed. New York: Dover. 1960, 33-66.
7. Shanti Narayan. A Course of Mathematical Analysis, S. Chand & Co., New Delhi
8. Goldberg RR. Real analysis, Oxford & IBH publishing Co., New Delhi, 1970.
9. Somasundaram D, Choudhary B. A First Course in Mathematical Analysis, Narosa Publishing House, New Delhi, 1997.
10. Apostol TM. Mathematical Analysis, Narosa Publishing House, New Delhi, 1985.