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Exploring B-Spline functions for numerical solution of mathematical problems

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Abstract
In the present paper, we describe Numerical solution of mathematical problems by strategically studying and researching the existing B-spline techniques. Here, we discussed the recursive relation for B-splines formulation by studying of B-splines of various degrees using different approaches. A comparative analysis between the resultant cubic spline applied on test equation with the existing cubic spline along with remarks is made in present study. A tabulated analysis of results of the present study is notable and at par with cubic spline.

Keywords: Spline functions, B-splines.

1. Introduction

The theory of spline functions is a very attractive field of approximation theory. Usually, a spline is a piecewise polynomial function defined in region D, such that there exists a decomposition of D into sub regions in each of which the function is a polynomial of some degree k. Also, the function, as a rule, is continuous in D, together with its derivatives of order up to $(k-1)$ [1-10]. Generally, the piecewise polynomial is considered, and $[a, b] \subset R$ is a finite interval. We introduce a set of partition $\Delta_n = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$, where x_i $(i = 0(1)\ n)$ are called nodes of the partition. The set of piecewise polynomial of degree k defined on a partition Δ_n is denoted by $S_k(\Delta_n)$ in each subinterval; $I_i = [x_{i-1}, x_i]$ is a k^{th} degree polynomial. Specifically, the type of bases B-spline for our purpose is considered, for which we only use the equidistance partition. Moreover, we extend the set of nodes by taking

$$h = \frac{b-a}{n}, x_0 = a \text{ and } x_i = x_0 + ih \text{ where } i = \pm 1, \pm 2, \pm 3, \ldots$$

Let $\{\Delta_n\}$ be a partition of $[a, b] \subset R$. A B-spline of degree k is a spline from $S_k(\Delta_n)$ with minimal support and the partition of unity holding.

The B-spline of degree k is denoted by $B_{i,k}(x)$, where $i \in Z$, and then we have the following properties:

1. Supp $(B_{i,k}) = \left[x_i, x_{i+k+1}\right]$.
2. $B_{i,k}(x) \geq 0. \forall x \in R$ (Non-negitivity).
3. $\sum_{i=-\infty}^{\infty} B_{i,k}(x) = 1, \forall x \in R$ (Partition of unity).

The next section explains the explicit definition of B-splines.

B-Spline

B-spline is a spline function that has minimal support with respect to given degree, smoothness and domain partition. The first reference to the world B-spline function ('B' refers to basis) in the field of mathematics was given by Schonberg in 1946, who described it as a smooth piecewise polynomial approximation and is short for basis spline. A B –spline is defined as a spline function that has minimal support with respect to a given degree, smoothness, and domain partition. The underlying core of the B-spline is its basis function. The defining feature of the basis function is knot sequence xi. Let X be a set of N+1 non decreasing real numbers $x_0 \le x_1 \le x_2 \le \ldots \le x_{n-1} \le x_n$. Here x_i^S are called knots , the set X is the knot sequence which represents the active area of real numbers line that defines the B-spline basis ,and the half –open interval $[x_i, x_{i+1})$ the i^{th} knot span. If the knots are equally spaced (*i e* $x_i - x_{i+1}$ . ., is a constant for $0 \le \imath \le N-1$), the knots vectors or the knot sequence is said to be uniform; otherwise, it is called non-uniform. Each B-spline function of degree *k* covers *k* □1 knots or *k* intervals. In early 1978's, a recurrence relation was independently established by Cox and Boor for the purpose of computing B-spline basis function. By applying the Leibniz' theorem, Boor was able to drive the following formula for m^{th} B-spline basis function of k^{th} degree in a recursive manner as follow:

$$B_{m,k}(x) = V_{m,k} B_{m,k-1}(x) + (1 - V_{m+1,k}) B_{m+1,k-1}(x) \quad (1)$$

Where $V_{m,k} = \left(\dfrac{x - x_m}{x_{m+k} - x_m} \right)$

This formula is known as Cox de- Boor recursion formula. Here $B_{m,k}(x)$ define a *mth* B-spline basis function of degree *k*, $\{x_i\}$ is non –decreasing set of real numbers also called as the knot sequence and *x* is a parameter variable. The recurrence relation starts with the first degree B-splines and builds the functions of successively higher orders. For degree $k \ge 1$ basis function $B_{m,k}(x)$ is a linear combination of two $(k-1)$ th degree basis function.

2. Methods

Derivation of B-spline functions

In this section we give an introduction of B-splines. The B-splines were so named because they formed a basis for the set of all splines. Throughout this section, we support that an infinite set of knots $\{x_i\}$ has been prescribed in such a way that

$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$$

$$\lim_{i \to \infty} x_i = \infty = -\lim_{i \to \infty} x_{-i}$$

The B-spline to be defined now depends on this set of knots.

Definition: Support of function *f* is defined as the set of points *x* when $f(x) \ne 0$.

B-spline of degree 0

For degree *k* =0, the basis function is just a step function. Thus, the zero degree B-spline is one of the simplest B-spline basis function and is given as

$$B_{m,0} = \begin{cases} 1, & x \in [x_m, x_{m+1}) \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Thus, a zero degree B-spline is equal to zero at all points expects on the half open interval $[x_m, x_{m+1})$.

B-Spline of degree 1

The expression for the first degree B- spline, also called as linear B-spline can be obtained using the Cox and Boor recursion formula given by (1).

Put k=1 in (1) and use the definition of zero degree B – spline. The formula of the first degree B-spline basis function can be given as

$$B_{m,1} = \begin{cases} \dfrac{x - x_m}{x_{m+1} - x_m}, & x \in [x_m, x_{m+1}) \\ \dfrac{x_{m+2} - x}{x_{m+2} - x_{m+1}}, & x \in [x_{m+1}, x_{m+2}) \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

The first degree B-spline is like a HAT or tent function which is non-zero for two knot spans $[x_m, x_{m+1})$ and $[x_{m+1}, x_{m+2})$ and can represented as

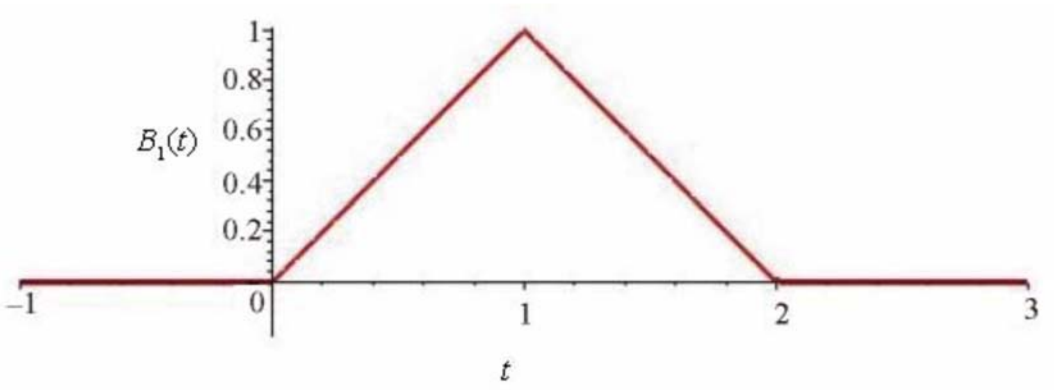

Fig 1: Uniform linear B-spline function (hat function).

B-Spline of degree 2(Quadratic)

The formula for the second degree B-spline also called as quadratic B-spline can be obtained by using the formula of linear B-spline basis function (3) and de- Boor recursion formula for k=2.

The formula for the second degree B-spline can be given as

$$B_{m,2} = \begin{cases} \dfrac{(x-x_m)^2}{(x_{m+2}-x_m)(x_{m+1}-x_m)}, & x \in [x_m, x_{m+1}) \\ \dfrac{(x-x_m)(x_{m+2}-x)}{(x_{m+2}-x_m)(x_{m+2}-x_{m+1})} + \dfrac{(x_{m+2}-x)(x-x_{m+1})}{(x_{m+3}-x_{m+1})(x_{m+2}-x_{m+1})}, & x \in [x_{m+1}, x_{m+2}) \\ \dfrac{(x_{m+3}-x)^2}{(x_{m+3}-x_{m+1})(x_{m+3}-x_{m+2})}, & x \in [x_{m+2}, x_{m+3}) \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

The second degree B-spline basis function is non-zero between three knot spans and can be represented as

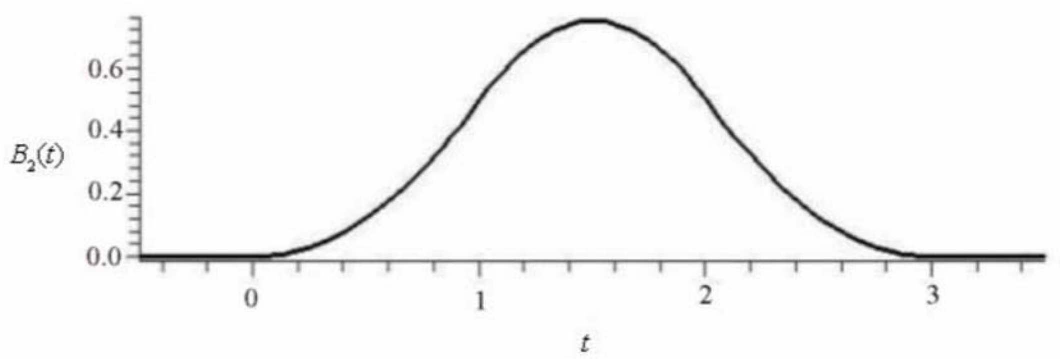

Fig 2: Uniform quadratic B-spline function.

Collocation Method Using B –Spline Basis Function

The collocation method together with B-spline approximation represents an economical alternative, since it is based on evaluating the accuracy of a differential equation at a finite set of collocation points .the issue that effect the effectiveness and accuracy of B-spline collocation method for solving differential equations include, which points to use for collocation, what degree of B-spline to use and what level of continuity to maintain.

Let us consider $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ as a uniform partition of the solution domain [*a*,*b*] by the knots x_m with step-length $h = x_{m+1} - x_m$, where $m = 0,1,2,...,n-1.$

Now to find the solution of differential equation using collocation method with B-spline basis function, the approximate solution *U*(*x*) can be assumed as a linear combination of basis functions as

$$U(x) = \sum_{j=m-k+2}^{m+k-2} c_j B_j(x) \quad (5)$$

Where, k is the degree of the B-spline, m is the number of nodes and c_m are the unknown constants to be determined from the boundary conditions and collocation from of the differential equation.

Let us now derive the approximate formula with the basis function of third, fourth and fifth degree B-spline.

B-Spline of degree 3

The third degree B-spline called as cubic B-spline basis function is given by formula

$$B_{m,3} = \frac{1}{h^3} \begin{cases} (x-x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}) \\ (x-x_{m-2})^3 - 4(x-x_{m-1})^3, & x \in [x_{m-1}, x_m) \\ (x_{m+2}-x)^3 - 4(x_{m+1}-x)^3, & x \in [x_m, x_{m+1}) \\ (x_{m+2}-x)^3, & x \in [x_{m+1}, x_{m+2}) \\ 0, & otherwise \end{cases} \quad (6)$$

This definition of cubic B-spline basis functions is given with x_m as the middle knot and equal number of knots on the two sides. The third degree B-spline is non-zero on four knot spans. From the definition given by (6), the value of $B_{m,3}(x)$ at the nodal points can be obtained. On differentiating with respect to *x* we can obtain the value of first and second derivatives of $B_{m,3}(x)$, the value of

$B_{m,3}(x)$ and its first derivatives at the nodal points can be tabulated as in Table 1.

Table 1: value of $B_{m,3}(x)$ and its first derivatives at the nodal points

	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$B_{m,3}(x)$	0	1	4	1	0
$B'_{m,3}(x)$	0	3/h	0	-3/h	0
$B''_{m,3}(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Now substituting k=3 in (5), we get

$$U(x)=\sum_{j=m-3+2}^{m+3-2} c_j B_j(x)$$

So the approximate solution can be written as

$$U(x)=\sum_{j=m-1}^{m+1} c_j B_j(x) \tag{7}$$

Without loss of generality equation (7) can be expressed as

$$U(x_m)=c_{m-1}B_{m-1}(x_m)+c_mB_m(x_m)+c_{m+1}B_{m+1}(x_m)$$

Or

$$U(x_m)=c_{m-1}B_m(x_{m+1})+c_mB_m(x_m)+c_{m+1}B_m(x_{m-1}) \tag{8}$$

As we are evaluated for cubic spline, so (8) can be written as

$$U(x_m)=c_{m-1}B_{m,3}(x_{m+1})+c_mB_{m,3}(x_m)+c_{m+1}B_{m,3}(x_{m-1}) \tag{9}$$

From here

$$U'(x_m)=c_{m-1}B'_{m,3}(x_{m+1})+c_mB'_{m,3}(x_m)+c_{m+1}B'_{m,3}(x_{m-1})$$

$$U''(x_m,t)=c_{m-1}B''_{m,3}(x_{m+1})+c_mB''_{m,3}(x_m)+c_{m+1}B''_{m,3}(x_{m-1})$$

On substituting values of $B_{m,3}(x)$ at the knots from Table 1.1, we get

$$U(x_m)=c_{m-1}+4c_m+c_{m+1}$$

$$hU'(x_m)=3(c_{m+1}-c_{m-1})$$

$$h^2U''(x_m,t)=6(c_{m-1}-2c_m+c_{m+1})$$

B-Spline of degree 4

The B-spline basis function of fourth degree also called as quartic B-spline is given by

$$B_{m,4}=\frac{1}{h^4}\begin{cases}(x-x_{m-2})^4, & x\in[x_{m-2},x_{m-1})\\ (x-x_{m-2})^4-5(x-x_{m-1})^4, & x\in[x_{m-1},x_m)\\ (x-x_{m-2})^4-5(x-x_{m-1})^4+10(x-x_m)^4, & x\in[x_m,x_{m+1})\\ (x_{m+3}-x)^4-5(x_{m+2}-x)^4, & x\in[x_{m+1},x_{m+2})\\ (x_{m+3}-x)^4, & x\in[x_{m+1},x_{m+2})\\ 0, & otherwise\end{cases} \tag{10}$$

This basis function is non zero on five knot spans. From definition given by (10) the value of $B_{m,4}(x)$ at the nodal points can be obtained on differentiating with respect to *x* we can obtain the value of its three derivatives in similar way. The value of $B_{m,4}(x)$ and its derivatives at the nodal points may be tabulated as in the table 2.

Table 2: values of $B_{m,4}(x)$ and its derivatives at nodal points

	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
$B_{m,4}(x)$	0	1	11	11	1	0
$B'_{m,4}(x)$	0	4/h	12/h	-12/h	-4/h	0
$B''_{m,4}(x)$	0	$12/h^2$	$-12/h^2$	$-12/h^2$	$12/h^2$	0
$B'''_{m,4}(x)$	0	$24/h^3$	$-72/h^3$	$72/h^3$	$-24/h^3$	0

Now substituting k=4 in (5), we get

$$U(x)=\sum_{j=m-4+2}^{m+4-2} c_j B_j(x)$$

So the approximate solution can be written as

$$U(x)=\sum_{j=m-2}^{m+2} c_j B_j(x) \tag{11}$$

Without loss of generality equation (11) can be expressed as

$$U(x_m)=c_{m-2}B_{m-2}(x_m)+c_{m-1}B_{m-1}(x_m)+c_mB_m(x_m)+c_{m+1}B_{m+1}(x_m)+c_{m+2}B_{m+2}(x_m)$$

Or

$$U(x_m)=c_{m-2}B_m(x_{m+2})+c_{m-1}B_m(x_{m+1})+c_mB_m(x_m)+c_{m+1}B_m(x_{m+1})+c_{m+2}B_m(x_{m+2})$$

As we are evaluating for quartic B-spline, so can be written as

$$U(x_m)=c_{m-2}B_{m,4}(x_{m+2})+c_{m-1}B_{m,4}(x_{m+1})+c_mB_{m,4}(x_m)+c_{m+1}B_{m,4}(x_{m-1})+c_{m+2}B_{m,4}(x_{m-2}) \quad (12)$$

So

$$U'(x_m)=c_{m-2}B'_{m,4}(x_{m+2})+c_{m-1}B'_{m,4}(x_{m+1})+c_mB'_{m,4}(x_m)+c_{m+1}B'_{m,4}(x_{m-1})+c_{m+2}B'_{m,4}(x_{m-2})$$

On substituting values of $B_{m,4}(x)$ at the knots from Table 1.2, we get

$$\begin{aligned}
U(x_m)&=c_{m-2}+11c_{m-1}+11c_m+c_{m+1}\\
hU'(x_m)&=4(-c_{m-2}-3c_{m-1}+3c_m+c_{m+1})\\
h^2U''(x_m,t)&=12(c_{m-2}-c_{m-1}-c_m+c_{m+1})\\
h^3U'''(x_m,t)&=24(-c_{m-2}+3c_{m-1}-3c_m+c_{m+1})
\end{aligned} \quad (13)$$

B-Spline of degree 5

The fifth degree B-spline, also called quintic B-spline basis function is given by formula

$$B_{m,4}=\frac{1}{h^4}\begin{cases}(x-x_{m-3})^5, & x\in[x_{m-3},x_{m-2})\\ (x-x_{m-3})^5-6(x-x_{m-2})^5, & x\in[x_{m-2},x_{m-1})\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5, & x\in[x_{m-1},x_m)\\ (x_{m+3}-x)^5-6(x_{m+2}-x)^5+15(x_{m+1}-x)^5, & x\in[x_m,x_{m+1})\\ (x_{m+3}-x)^5-6(x_{m+2}-x)^5, & x\in[x_{m+1},x_{m+2})\\ (x_{m+3}-x)^5, & x\in[x_{m+2},x_{m+3})\\ 0, & otherwise\end{cases} \quad (14)$$

This basis function is non zero on six knot spans. From definition given by (14) the value of $B_{m,5}(x)$ at the nodal points can be obtained and on differentiating with respect to x the value of its four derivatives can be obtained in similar way. The value of $B_{m,5}(x)$ and its derivatives at the nodal points may be tabulated as in the table 3.

Table 3: Value of $B_{m,5}(x)$ for quintic B-spline and its derivatives at the nodal points.

	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
$B_{m,5}(x)$	0	1	26	66	26	1	0
$B'_{m,5}(x)$	0	5/h	50/h	0	-50/h	-5/h	0
$B''_{m,5}(x)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
$B'''_{m,5}(x)$	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0
${B^{iv}}_{m,5}(x)$	0	$120/h^4$	$-480/h^4$	$720/h^4$	$-480/h^4$	$120/h^4$	0

Now substituting k=5 in (5), we get

$$U(x)=\sum_{j=m-5+2}^{m+5-2} c_j B_j(x)$$

So the approximate solution can be written as

$$U(x)=\sum_{j=m-3}^{m+3} c_j B_j(x) \tag{15}$$

Without loss of generality equation (15) can be expressed as

$$U(x_m)=c_{m-3}B_{m-3}(x_m)+c_{m-2}B_{m-2}(x_m)+c_{m-1}B_{m-1}(x_m)+c_mB_m(x_m)$$
$$+c_{m+1}B_{m+1}(x_m)+c_{m+2}B_{m+2}(x_m)+c_{m+3}B_{m+3}(x_m)$$

Or

$$U(x_m)=c_{m-3}B_m(x_m)+c_{m-2}B_m(x_m)+c_{m-1}B_m(x_m)+c_mB_m(x_m)$$
$$+c_{m+1}B_m(x_{m-1})+c_{m+2}B_m(x_{m-2})+c_{m+3}B_m(x_{m-3})$$

As we are evaluating for quartic B-spline, so can be written as

$$U(x_m)=c_{m-3}B_{m,5}(x_{m+3})+c_{m-2}B_{m,5}(x_{m+2})+c_{m-1}B_{m,5}(x_{m+1})+c_mB_{m,5}(x_m)$$
$$+c_{m+1}B_{m,5}(x_{m-1})+c_{m+2}B_{m,5}(x_{m-2})+c_{m+3}B_{m,5}(x_{m+3}) \tag{16}$$

On substituting values of $B_{m,5}(x)$ at the knots from Table 1.3, we get

$$U(x_m)=c_{m-2}+26c_{m-1}+66c_m+26c_{m+1}+c_{m+2}$$
$$hU'(x_m)=5(-c_{m-2}-10c_{m-1}+10c_{m+1}+c_{m+2}) \tag{17}$$
$$h^2U''(x_m)=20(c_{m-2}+2c_{m-1}-6c_m+2c_{m+1}+c_{m+2})$$
$$h^3U'''(x_m)=60(-c_{m-2}+2c_{m-1}-2c_{m+1}+c_{m+2})$$
$$h^4U^{iv}(x_m)=120(c_{m-2}-4c_{m-1}+6c_m-4c_{m+1}+c_{m+2})$$

Properties of B-Spline Basis Function

Some of the important properties of the B-spline basis functions are as follows:

1. $B_{m,k}(x)$ is a non-zero polynomial on $[x_m, x_{m+k+1})$ for degree $k \geq 0$.

2. On any span $[x_m, x_{m+1})$ at most k +1 basis functions of degree k are non-zero $B_{m-k,k}(x), B_{m-k+1,k}(x), B_{m-k+2,k}(x), \ldots,$ and $B_{m,k}(x)$.

3. Non negativity, for all *m*, *k* and x, $B_{m,k}(x)$ is non-negative in the interval $[x_m, x_{m+1})$. The closed interval is called the support of $B_{m,k}(x)$.

4. Local knots, the m^{th} B-spline $B_{m,k}(x)$ depends only on the knots $x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+k+1}$.

5. Local Support, if x is outside the interval $[x_m, x_{m+k+1})$ then $B_{m,k}(x)=0$.

Local support property indicates that each segment of a B-spline curve is influenced by only *k* control points or each control point affects only *k* curve segments.

Numerical Results

In order to demonstrate the application of the proposed B-spline method, we have solved the following problems whose analytical solutions are known to us.

Example: Consider the following equations

$$u''(x)+xu(x)+xv(x)=f_1(x),$$
$$v''(x)+2xv(x)+2xu(x)=f_2(x),$$

subject to the boundary conditions *u(0)=u(1)=0, v(0)=v(1)=0* where *0<x<1,* $f_1(x)=2$ and $f_2(x)=-2$. The exact solutions *u(x,)* *v(x)* are x^2-x and $x-x^2$, respectively. The observed maximum absolute errors for various values of *n* are given in Table 4. The numerical results are illustrated in Fig.3 and Fig.4.

Table 4: The maximum absolute errors

N	Absolute error u(x)	Absolute error v(x)
11	$3.41*10^{-15}$	$1.85*10^{-15}$
21	$1.89*10^{-3}$	$8.60*10^{-5}$
31	$4.74*10^{-4}$	$2.35*10^{-5}$
41	$2.10*10^{-4}$	$1.11*10^{-5}$

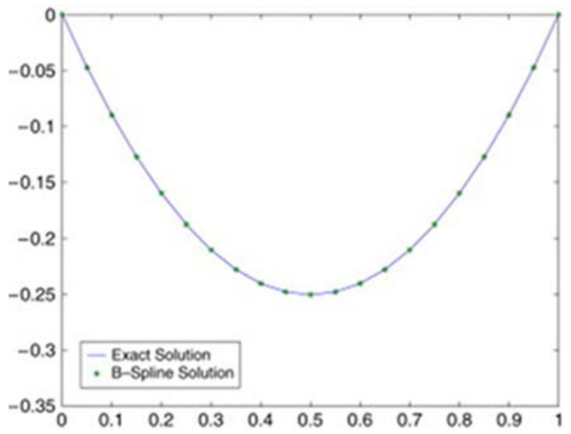

Fig 3: Results with

$$u(x) = x^2 - xu(x) = x^2 - x.$$

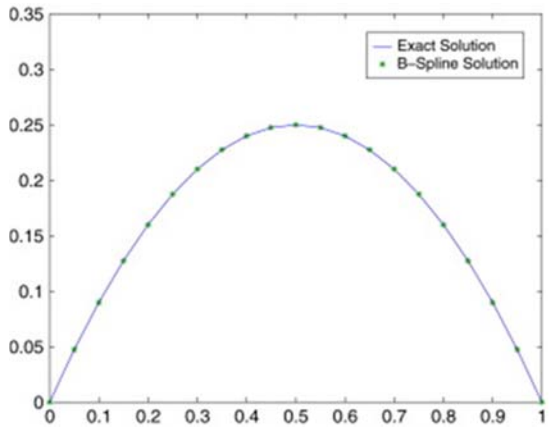

Fig 4: Results with

$$v(x) = x - x^2 v(x) = x - x^2.$$

3. Conclusions

These tables show that the results obtained by cubic B-spline are considerable and accurate with respect to the cubic spline.

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