On the Non -Homogeneous Sextic Equation \( x^4 + 2(x^2 - w)x^2y^2 + y^4 = z^4 \)


Abstract
We obtain infinitely many non-zero integer quadruples \((x, y, z, w)\) satisfying the non-homogeneous sextic equation with four unknowns \( x^4 + 2(x^2 - w)x^2y^2 + y^4 = z^4 \). Various interesting properties among the values of \( x, y, z \) and \( w \) are presented.

Keywords: sextic equation with four unknowns, integral solutions.

1. Introduction
The theory of diophantine equations offers a rich variety of fascinating problems \([1-4]\). Particularly, in \([5, 6]\), sextic equations with 3 unknowns are studied for their integral solutions. \([7, 9]\) analyse sextic equations with 4 unknowns for their non-zero integer solutions. This communication concerns with another non-zero sextic equation with 4 unknowns given by \( x^4 + 2(x^2 - w)x^2y^2 + y^4 = z^4 \). Infinitely many non-zero integer quadruples \((x, y, z, w)\) satisfying the above equation are obtained. Various interesting properties among the values of \( x, y, z \) and \( w \) are presented.

2. Method of Analysis:
The diophantine equation representing the sextic equation with four unknowns under consideration is given by
\[
x^4 + 2(x^2 - w)x^2y^2 + y^4 = z^4
\]
(1)

(1) Is written as the system of double equations
\[
w^2 = 2y^2 + 1
\]
(2)
\[
z^2 = wx^2 - y^2
\]
(3)

(2) Is the well-known pellian equation whose general solution is given by
\[
y_n = \frac{1}{2}\sqrt{2}[(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}]
\]
(4)
\[
w_n = \frac{1}{2}((3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1})
\]
(5)

Substitution of (4) & (5) in (3) gives
\[
z_n = w_n x^2 - y_n^2, n = 0, 1, 2, 3, ...
\]
(6)

As it is not possible to obtain a general solution pattern of (6), we have to take particular values of \( n \) for getting a general form of solution of (6) corresponding to each value of \( n \).

For illustration, the choice \( n=1 \) in (4) and (5) gives \( y_1 = 12, w_1 = 17 \) and thus (6) becomes
\[
z = 17x^2 - 12z^2 = z^2
\]
(7)

Initial solution is \( z_0 = 3, x_0 = 3 \)

Consider the pellian \( z^2 = 17x^2 + 1 \)

Initial solution is \( z_0 = 33, x_0 = 8 \)

The general integral solution \( (\tilde{x}_n, \tilde{z}_n) \) of (8) is obtained as
\[
\tilde{x}_n = \frac{1}{2}\sqrt{7}[(33 + 8\sqrt{7})^{n+1} + (33 - 8\sqrt{7})^{n+1}]
\]
(10)
\[
\tilde{z}_n = \frac{1}{2}\sqrt{7}[(33 + 8\sqrt{7})^{n+1} - (33 - 8\sqrt{7})^{n+1}]
\]
(11)

The solution of (7) is obtained as
\[
z_{n+1} = z_0\tilde{z}_n + D_{n+1}\tilde{x}_n = 3z_n + 51x_n
\]
\[
x_{n+1} = z_0\tilde{x}_n + x_0\tilde{z}_n = 3\tilde{x}_n + 32z_n
\]

Thus the quadruples \((\tilde{x}_n, 12, \tilde{z}_n, 17)\) represent the non-zero distinct integer solutions of (1).
The recurrence relations among the solutions \((x_n, z_n)\) of (1) are given by

\[ x_{n+3} - 66x_{n+2} + x_{n+1} = 0, x_0 = 3, x_1 = 123, x_2 = 8115 \]
\[ z_{n+3} - 66z_{n+2} - z_{n+1} = 0, z_0 = 3, z_1 = 507, z_2 = 33465 \]

2.1 Properties:
1. \(17x_{n+2} - z_{n+2} + 48\) is a nasty number
2. \(9[17x_{n+1} - z_{n+1} + 3(17x_{n+1} - z_{n+1})]\) is a cubic integer.
3. \(54[17x_{n+4} - z_{n+4} + 4(17x_{n+2} - z_{n+2}) + 144]\) is a biquadratic integer
4. \(17(x_{n+1} - z_{n+1})^2 - 24(17x_{n+2} - z_{n+2}) + 1152 = 0\)
5. \(17x_{2n+2} - z_{2n+2} - 17(z_{n+1} - x_{n+1})^2 = 0\) (triangular number of rank 47)
6. Define: \(X = 17x_{2n+2} - z_{2n+2} + 48, \ Y = 17x_{n+1} - z_{n+1}\)
It is to be noted that the pair \((X, Y)\) satisfies the parabola \(Y^2 = 24X\)

3. Conclusion:
In conclusion, by taking other values of \(n\) in (6) and following a similar analysis, infinitely many integer quadruples satisfying (1) are determined

4. References: