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### Kulkarni P. R

P. G. Department of  
Mathematics, N.E.S. Science  
College, Nanded. (M.S.),  
India

### Borkar V. C

Department of Mathematics  
and statistics, Yeshwant  
Mahavidyalaya, Nanded.  
(M.S.), India

## Period three cycle and chaos in a dynamical system

**Kulkarni P. R, Borkar V. C**

### Abstract

The mathematical modeling of many phenomenon in the fields of physics, chemistry, astronomy, astrophysics, ecology, economics and many more has been done by scientists and the dynamics of such models has been studied all over the world. The modeling of the 'Predator and Prey' phenomenon in ecosystem in terms of the logistic family  $F_{\mu}(x) = \mu x(1-x)$  has played an important role in the development of the subject of dynamical system and chaos. The chaotic behavior of many families of mappings like the Tent family, Quadratic family, the logistic family, etc. has been already proved by many authors. The chaotic nature, in the sense of Devaney R. L., of the family of mappings  $f_c(x) = x^2 - x + c$  through the period doubling cascade and using the concept of topological conjugacy has already been proved by Kulkarni P. R. and Borkar V. C. In this paper, we have proved the existence of the period three cycle in the family of mappings

$f_c(x) = x^2 - x + c$  for  $c = -1.5$  and thereby proved that this mapping exhibits chaos.

**Keywords:** chaos, dynamical system, fixed points, periodic points, period three, topological conjugacy

### 1. Introduction

Non-linear dynamics has been a topic of interest since last twenty years for the mathematicians, physicists, biologists, all over the world because of its wide range of applications in almost all branches of science and social sciences. Mathematicians have been trying to give more and more accurate mathematical model of a physical phenomenon and predict the changes that will take place over time. Dynamical systems and specially chaotic dynamics has been a topic of more interest for scientists.

#### 1.1 Dynamical System

A dynamical system consists of two parts: a state vector  $x \in R^n$ , which is a list of numbers and may change as the time passes, and a function  $f: R^n \rightarrow R^n$ , where the set  $R^n$  is called as the set of states or the state space. By means of the function  $f$ , one can determine the state of the vector at any position from the current state. This rule is described by the function  $f$ . Most of the mathematicians study two types of dynamical systems viz. discrete dynamical system and continuous dynamical system. The exact definitions and a number of examples are given by [1], [2], [3], [4] and [5]. In this paper, we will consider only one dimensional discrete dynamical systems.

Basic ideas in the field of dynamical systems have been introduced in the early text of the references provided in this paper; however, for the convenience of the reader, we introduce some of them in brief.

#### 1.2 Iterations of a function

For a given dynamical system  $f: S \rightarrow S$ ,  $S \subseteq R$ , iterations of the function  $f$  means the compositions of  $f$  with itself.

In general, the  $k^{\text{th}}$  iteration of  $f$  at a point  $x$  is the  $k$  times composition of  $f$  with itself at the point  $x$ , denoted by  $f^k(x)$ . For more details, refer [1] [4], [5], [7].

#### 1.3 Fixed Points and Periodic Points

A point  $x$  is said to be a fixed point of a function  $f$  if  $f(x) = x$ . If  $x$  is a fixed point of  $f$ , then  $f^n(x) = x$  for all  $n \in Z^+$ , where is  $Z^+$  the set of positive integers.

A point  $x$  is said to be a periodic point with period  $n$  if there exists a positive integer  $n$  such that  $f^n(x) = x$ . Clearly, if  $x$  is periodic with period  $n$ , then it is periodic with period  $2n, 3n, 4n, \dots$ . The smallest  $n$ , in this case, is called as the prime period of the orbit. Note that  $x$  is a periodic point with period  $n$  of  $f$  if it is a fixed point of  $f^n$ . Refer [1] [4], [5], [7] for more details.

### Correspondence:

### Kulkarni P. R

P. G. Department of  
Mathematics, N.E.S. Science  
College, Nanded. (M.S.),  
India

### 1.4 Attracting and Repelling Fixed Points

Let  $p$  be a fixed point of a dynamical system  $f: S \rightarrow S, S \subseteq R$ .

- (1) We say that  $p$  is an attracting fixed point or a sink of  $f$  if there is some neighborhood of  $p$  such that all points in this neighborhood are attracted towards  $p$ . In other words,  $p$  is a sink if there exists an epsilon neighborhood  $N_\epsilon(p) = \{x \in S: |x - p| < \epsilon\}$  such that  $\lim_{n \rightarrow \infty} f^n(x) = p$  for all  $x \in N_\epsilon(p)$ .
- (2) We say that  $p$  is a repelling fixed point or a source of  $f$  if there is some neighborhood  $N_\epsilon(p)$  of  $p$  such that each  $x$  in  $N_\epsilon(p)$  except for  $p$  maps outside of  $N_\epsilon(p)$ . In other words,  $p$  is a source if there exists an epsilon neighborhood such that  $|f^n(x) - p| > \epsilon$  for infinitely many values of positive integers  $n$ .

### 1.5 Hyperbolic Periodic Points

A periodic point  $p$  of a mapping  $f$  with prime period  $n$  is said to be hyperbolic if  $|(f^n)'(p)| \neq 1$ .

### 1.6 Attracting and Repelling Periodic Point

Let  $p$  be a periodic point of period  $n$  of a function  $f$ . Then  $p$  is said to be an attracting periodic point or a repelling periodic point according as it is an attracting or a repelling fixed point of the  $n^{\text{th}}$  iterate  $f^n$ .

### 1.7 Neutral Fixed Point

A fixed point  $p$  of a differentiable function  $f$  is said to be a neutral fixed point if  $|f'(p)| = 1$ .

### 1.8 A review of the results

Let us recall some of the theorems that have been already proved. For the proof, one can look in to the references provided.

#### 1.9 Theorem

Let  $f: [a, b] \rightarrow R$  be a differentiable function, where  $f'$  be continuous and  $p$  be a periodic point of  $f$  with period  $n$ . Then the periodic orbit of  $p$  is attracting or repelling according as  $|(f^n)'(p)| < 1$  or  $|(f^n)'(p)| > 1$ .

#### 1.10 Theorem

Let  $f: [a, b] \rightarrow R$  be a differentiable function, where  $f'$  be continuous and  $p$  be a hyperbolic fixed point of  $f$ . If  $|f'(p)| < 1$ , then  $p$  is an attracting fixed point of  $f$ .

#### 1.11 Theorem

Let  $f: [a, b] \rightarrow R$  be a differentiable function, where  $f'$  be continuous and  $p$  be a hyperbolic fixed point of  $f$ . If  $|f'(p)| > 1$ , then  $p$  is a repelling fixed point of  $f$ .

#### 1.12 Theorem

Let  $p$  be a neutral fixed point of a function  $f$ .

- (i) If  $f''(p) > 0$ , then  $p$  is weakly attracting from the left and weakly repelling from the right.
- (ii) If  $f''(p) < 0$ , then  $p$  is weakly repelling from the left and weakly attracting from the right.

Let  $p$  be a neutral fixed point of  $f$  with  $f''(p) = 0$ .

- (iii) If  $f'''(p) > 0$  then  $p$  is weakly repelling.
- (iv) If  $f'''(p) < 0$ , then  $p$  is weakly attracting.

## 2. Chaotic Dynamical Systems

We use the word *chaos* in our daily life as an indication of confusion or ambiguity or randomness or uncertainty. Chaotic dynamics have been the topic of key interest

amongst the scientists as the chaos is observed in nature most of the times. While defining chaos, many approaches have been made by different authors including measure theoretic notions, topological concepts, etc. We will use the most widely used and accepted definition of chaos as given by Devaney, R. L. [4].

### 2.1 Chaos

Let  $V$  be a set. A mapping  $F: V \rightarrow V$  is said to be chaotic on  $V$  if

1.  $F$  has sensitive dependence on initial conditions.
2.  $F$  is topologically transitive.
3. periodic orbits are dense in  $V$ .

For details of the notions used in this definition, the reader is advised to refer [4], [7],[9].

### 2.2 Examples of chaotic mappings

Some examples of the mappings showing chaotic behavior are as follows. The chaotic nature of the first three mappings has been proved in the book by Devaney[4].

1. Let  $S^1$  denote the unit circle in plane. The mapping  $f: S^1 \rightarrow S^1$  defined by  $f(\theta) = 2\theta$  is chaotic, where a point in  $S^1$  is denoted by its angle  $\theta$  measured in radians.
2. The logistic family  $F_\mu(x) = \mu x(1-x)$  is chaotic on the Cantor set  $\Lambda$  for  $\mu > 2 + \sqrt{5}$ . On a comparatively larger set  $I = [0, 1]$ , the mapping  $F_\mu(x) = \mu x(1-x)$  is chaotic for  $\mu = 4$ .
3. The tent mapping  $T_2$  defined by

$$T_2(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{for } 1/2 < x \leq 1 \end{cases}$$

is chaotic on the interval  $[0, 1]$ .

4. The chaos in the family of mappings  $f_c(x) = x^2 - x + c$  for the values of  $c > -1/2$  has been explored through a period doubling phenomenon by Kulkarni P. R. and Borkar V. C. [7] Moreover, considering the topological aspects of the dynamical systems, Kulkarni P. R. and Borkar V. C. [9] has proved the chaos in the same mapping by proving that it is topologically conjugate to the to the shift map  $\sigma$  on  $\Sigma_2$  via the itinerary  $S$ .

### 2.3 Period three and chaos

Now we will study the importance of the existence of period three in a dynamical system. The chaotic nature of a dynamical system is also guaranteed by the existence of the period three point. First we will state the period three theorem which was proved by Tien-Yien Li and James A. Yorke [6] in 1975.

### 2.4 Theorem (The Period Three Theorem)

Let  $f: R \rightarrow R$  be a continuous function. If  $f$  has a periodic point of period three, then  $f$  has periodic points of all other periods.

For the proof of this theorem the reader can also refer the book by Devaney [4]. A generalization of this theorem is known as the Sarkovskii's theorem. Sarkovskii's theorem was proved by Sarkovskii in 1964. This theorem explains the order of the existence of the periodic points *i.e.* which period imply which other periods for continuous dynamical systems. Before stating the Sarkovskii's theorem, we need to understand the Sarkovskii ordering of natural numbers. This ordering is as follows.

- 3, 5, 7, 9, ..., (list of all odd numbers except 1)
- 2.3, 2.5, 2.7, ..., (two times the odd naturals)
- 2<sup>2</sup>.3, 2<sup>2</sup>.5, 2<sup>2</sup>.7, ..., (2<sup>2</sup> times the odd naturals)
- 2<sup>3</sup>.3, 2<sup>3</sup>.5, 2<sup>3</sup>.7, ..., (2<sup>3</sup> times the odd naturals)
- 2<sup>n</sup>, 2<sup>n-1</sup>, 2<sup>n-2</sup>, ..., 2<sup>3</sup>, 2<sup>2</sup>, 2, 1.

This ordering by Sarkovskii exhausts the natural numbers. Now we state the Sarkovskii's theorem as follows.

**2.5 Theorem (Sarkovskii's Theorem)**

If a continuous function  $f : R \rightarrow R$  has a point of period  $n$ , where  $n$  precedes  $k$  in Sarkovskii's ordering of natural numbers, then  $f$  has a periodic point of period  $k$ .

The proof of this theorem can be studied from [4], [5] and [6]. As the number 3 appears at the first in the Sarkovskii's ordering of natural numbers, if a function has a point of period 3, then there exists periodic points of all periods. This is nothing but the Period Three Theorem as we stated earlier. Since the powers of 2 appear in the last line of the Sarkovskii's ordering of natural numbers, we conclude that if a function  $f$  has finitely many periodic points, then they all must be some powers of 2. On the contrary, if  $f$  has a periodic point of period which is not a power of 2, then  $f$  must have infinitely many period points. Sarkovskii's has been proved only for one dimensional dynamical systems. The theorem does not hold for higher dimensional systems. The converse of Sarkovskii's theorem is also true, which can be stated as follows

**2.6 Theorem (Converse of Sarkovskii's Theorem)**

Let  $k$  and  $n$  be natural numbers. If  $k$  precedes  $n$  in the Sarkovskii's ordering of natural numbers, then there exists a continuous mapping  $f : R \rightarrow R$  which has a periodic point of period  $n$  but no periodic point of period  $k$ .

**3. Topological Conjugacy**

The topological properties like existence of dense periodic orbit and topological transitivity [4] are inherited from one dynamical system to another through a bijective mapping called topological conjugacy. It plays an important role in studying important properties of an unknown dynamical system by comparing those with a known dynamical system. We define topological conjugacy as follows.

**3.1 Topological conjugacy**

Let  $f : A \rightarrow A$  and  $g : B \rightarrow B$  be two mappings. Then  $f$  is said to be topologically conjugate to  $g$  if there exists a homeomorphism  $h : A \rightarrow B$  such that  $h \circ f \circ h^{-1} = g$  or what amounts to the same thing, if  $h \circ f = g \circ h$ .

As  $h$  is a homeomorphism, it follows that if  $f$  is topologically conjugate to  $g$ , then  $g$  is also topologically conjugate to  $f$ . In this case, the homeomorphism  $h$  is called as a topological conjugacy between  $f$  and  $g$ , or we say that  $f$  and  $g$  are conjugate via the mapping  $h$ . The following results concerning the topological conjugacy are proved by Kulkarni P. R. and Borkar V. C.[9] We state these results here for reference.

**3.2 Theorem**

If  $f$  and  $g$  are topologically conjugate via mapping  $h$ , and if  $p$  is a fixed point of  $f$ , then  $h(p)$  is a fixed point of  $g$ .

**3.3 Theorem**

If  $f$  and  $g$  are topologically conjugate via mapping  $h$ , then  $f^n = g^n \circ h$ . Thus, if  $f$  and  $g$  are conjugate, the periodic points are carried into periodic points of the same period under conjugacy.

**3.4 Theorem**

Let  $f : A \rightarrow A$  and  $g : B \rightarrow B$  be topologically conjugate via a mapping  $h : A \rightarrow B$ . Then  $f$  is transitive if and only if  $g$  is

transitive. That is, topological conjugacy preserves transitivity.

**3.5 Theorem**

If  $h : A \rightarrow B$  is an onto continuous mapping, then the image under  $h$  of a set dense in  $A$  is a set dense in  $B$ .

**3.6 Theorem**

Let  $Per_n(f)$  denote the set of periodic points of period  $n$  of the mapping  $f$ . Let  $f$  and  $g$  be topologically conjugate. Then  $Per_n(f)$  is dense if and only if  $Per_n(g)$  is dense. From these theorems and the definition of chaos, it follows that if two mappings are topologically conjugate, then the number and nature of their fixed and periodic points is the same. We summarize this as the following.

**3.7 Theorem**

Let  $f$  and  $g$  be topologically conjugate. Then  $f$  is chaotic if and only if  $g$  is chaotic.

**4. Existence of Period Three and Chaos in  $f_c(x) = x^2 - x + c$**

Now we will obtain a period three point for the mapping  $f_c(x) = x^2 - x + c$  and thereby prove that  $f_c(x) = x^2 - x + c$  is chaotic on  $R$  for  $c = -1.25$ . As a first attempt in this direction, we will find period three points for the mapping  $T_2$  defined by

$$T_2(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{for } 1/2 < x \leq 1 \end{cases}$$

It is clear that  $\{2/7, 4/7, 6/7\}$  is a 3-cycle for  $T_2$  since

$$T_2\left(\frac{2}{7}\right) = \frac{4}{7}, T_2^2\left(\frac{2}{7}\right) = T_2\left(\frac{4}{7}\right) = \frac{6}{7}, T_2^3\left(\frac{2}{7}\right) = \frac{2}{7}$$

Now we will prove that the mapping  $T_2$  is topologically conjugate to the mapping  $F_4(x) = 4x(1-x)$  via the mapping  $h(x) = \sin^2\left(\frac{\pi x}{2}\right)$ . We obtain the composition  $F_4 \circ h$  as

$$(F_4 \circ h)(x) = F_4(h(x)) = F_4\left(\sin^2\left(\frac{\pi x}{2}\right)\right) = 4 \cdot \sin^2\left(\frac{\pi x}{2}\right) \left(1 - \sin^2\left(\frac{\pi x}{2}\right)\right) = \sin^2(\pi x) \dots (1)$$

$$\text{If } 0 \leq x \leq \frac{1}{2}, \text{ then } \left(h \circ T_2\right)(x) = h(T_2(x)) = h(2x) = \sin^2(\pi x) \dots (2)$$

$$\text{and if } \frac{1}{2} < x \leq 1, \text{ then } \left(h \circ T_2\right)(x) = h(2(1-x)) = \sin^2(\pi - \pi x) = \sin^2(\pi x) \dots (3)$$

Thus from Eq.(1), Eq.(2), and Eq.(3), we have  $F_4 \circ h = h \circ T_2$  so that  $T_2$  is topologically conjugate to the mapping  $F_4$  via the mapping  $h(x) = \sin^2\left(\frac{\pi x}{2}\right)$ . As  $\{2/7, 4/7, 6/7\}$  is a 3-cycle for  $T_2$ , by theorem 3.2, it follows that  $\{\sin^2\left(\frac{\pi}{7}\right), \sin^2\left(\frac{2\pi}{7}\right), \sin^2\left(\frac{3\pi}{7}\right)\}$  is a 3-cycle for  $F_4$ . Similarly we can prove that the mapping  $F_4(x) = 4x(1-x)$  is conjugate to the mapping  $f_{-1.25}(x) = x^2 - x - 1.25$  via the mapping  $H(x) = -4x + 2.5$ .

Hence a 3-cycle for  $f_{-1.25}(x) = x^2 - x - 1.25$  is  $\{2.5 - 4 \sin^2\left(\frac{\pi}{7}\right), 2.5 - 4 \sin^2\left(\frac{2\pi}{7}\right), 2.5 - 4 \sin^2\left(\frac{3\pi}{7}\right)\}$ .

An approximate value of this 3-cycle is  $\{1.746979603717467, 0.054958132087371, -1.3019377358048\}$ .

The iterates of a function can be studied by means of the graphical analysis. The orbit diagram for the function  $f_{-1.25}(x) = x^2 - x - 1.25$  is as shown in the following Figure1.

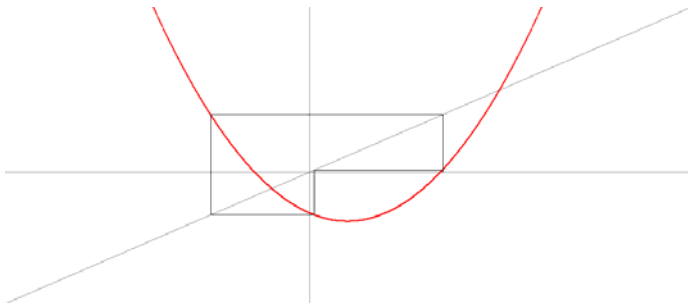


Fig 1:

A bifurcation diagram (Refer [4], [5], [7], [9]) exhibits the transition to chaos through successive period-doubling bifurcations as the parameter value is varied in one parameter family of mappings. For the one parameter family of mappings  $f_c(x) = x^2 - x + c$ , all the interesting dynamics occur in the interval  $-2 \leq c \leq 1$ . We divide the parameter range  $[-2, 2]$  into a number of specified subdivisions and the orbits are computed and plotted using the initial condition  $x_0 = 0$  and obtain the bifurcation diagram as shown in the Figure 2.

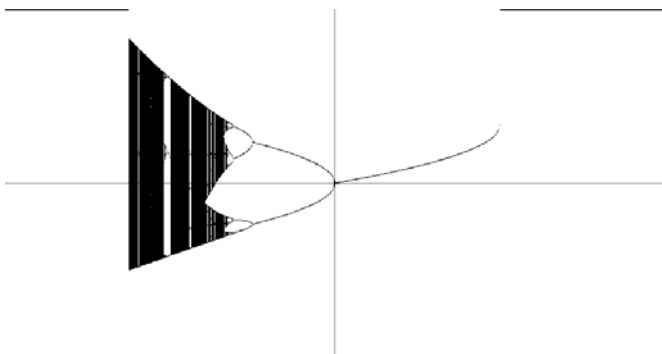


Fig 2:

By Sarkovskii's theorem, the mapping  $f_c(x) = x^2 - x + c$  is chaotic on  $R$  for  $c = -1.25$ .

**5. Conclusion**

The chaos in the behavior of a one parameter family of mappings can be noted by observing the bifurcation diagram only however, it becomes very difficult to find out at what value of the parameter, the periodic orbit of a particular period comes in to picture of the bifurcation diagram. We have obtained the value of  $c$  for which the family of mappings  $f_c(x) = x^2 - x + c$  has a period-3 orbit and thus proved that it is chaotic.

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