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Some general indices of certain *R-Crown* Molecular graphs

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Abstract
Some chemical indices have been invented in theoretical chemistry, such as geometric-arithmetic index and Zagreb index. In this paper, we determine these indices of certain *r*-crown molecular graphs.

Keywords: molecular graph, general third geometric-arithmetic index, generalized Zagreb index, general harmonic index, general second geometric-arithmetic index, general Co-PI index

1. Introduction

Geometric-arithmetic index, Randic index and other chemical indices are introduced to reflect certain structural features of organic molecules (See Yan et al., [1], Gao et al., [2], Gao and Shi [3], Gao and Wang [4], Xi and Gao [5-6], Xi et al., [7], Gao et al., [8] for more detail).

Let *e*=*uv* be an edge of the molecular graph *G*. The number of edges of *G* whose distance to the vertex *u* is smaller than the distance to the vertex *v* is denoted by $m_u(e)$. Analogously, $m_v(e)$ is the number of edges of *G* whose distance to the vertex *v* is smaller than the distance to the vertex *u*. Note that edges equidistant to *u* and *v* are not counted. Zhou et al., [9] proposed a third class of geometric-arithmetic index:

$$GA_3(G) = \sum_{e=uv\in E(G)} \frac{2\sqrt{m_u(e)m_v(e)}}{m_u(e)+m_v(e)}.$$

Recently, Gao [10] raised new version of third geometric-arithmetic index (i.e., general third geometric-arithmetic index):

$$GA_3^k(G) = \sum_{e=uv\in E(G)} \left(\frac{2\sqrt{m_u(e)m_v(e)}}{m_u(e)+m_v(e)}\right)^k.$$

Azari and Iranmanesh [11] introduced generalized Zagreb index of molecular graph *G* stated as:

$$M_{\{t_1,t_2\}}(G) = \sum_{uv\in E(G)} (d(u)^{t_1}d(v)^{t_2} + d(u)^{t_2}d(v)^{t_1}),$$

where t_1 and t_2 are arbitrary non-negative integers.

Very recently, Yan et al., [12] raised general version of harmonic index, the general harmonic index of molecular graph *G* is given as:

$$H_k(G) = \sum_{uv\in E(G)} \left(\frac{2}{d(u)+d(v)}\right)^k,$$

where *k* is a real number. In what follows, we always assume *k* is a real number.

Gao and Wang [13] raised general second geometric-arithmetic index as:

$$GA_2^k(G) = \sum_{uv\in E(G)} \left(\frac{2\sqrt{n(u)n(v)}}{n(u)+n(v)}\right)^k.$$

Hasani et al., [14] introduced Co-PI index as

$$Co-PI_v(G) = \sum_{e=uv\in E(G)} |n(u)-n(v)|.$$

It is proved that $Co-PI_v(G) = \sum_{e=uv\in E(G)} |T(u)-T(v)|$, where $T(u) = T_G(u) =$

$\sum_{v\in V} d_G(u,v)$.

Also, Gao and Wang [13] raised general Co-PI index as:

$$Co-PI_v^k(G) = \sum_{e=uv\in E(G)} |n(u)-n(v)|^k.$$

In this paper, we present the general third geometric-arithmetic index, generalized Zagreb index, general harmonic index, general second geometric-arithmetic index and general Co-PI index of $I_r(F_n)$, $I_r(W_n)$, $I_r(\tilde{F}_n)$ and $I_r(\tilde{W}_n)$.

2. General third geometric-arithmetic index

Theorem 1. $GA_3^k(I_r(F_n)) = 2(\frac{2\sqrt{(2n+nr-r-4)(r+1)}}{2n+nr-3})^k + 2(\frac{2\sqrt{(2n+nr-2r-4)(r+2)}}{2n+nr-r-2})^k + (n-4)(\frac{2\sqrt{(2n+nr-2r-5)(r+2)}}{2n+nr-r-3})^k + 2(\frac{2\sqrt{(r+1)(2r+3)}}{3r+4})^k + 2(\frac{2\sqrt{(2r+2)(2r+3)}}{4r+5})^k + (n-5) + r(n+1)(\frac{2\sqrt{2n+r+nr-2}}{2n+r+nr-1})^k$.

Proof. Let $P_n=v_1v_2\ldots v_n$ and the r hanging vertices of v_i be v_i^1, v_i^2, …, v_i^r ($1\le i\le n$). Let v be a vertex in F_n beside P_n, and the r hanging vertices of v be v^1, v^2, …, v^r.

Using the definition of general third geometric-arithmetic index, we have

$$GA_3^k(I_r(F_n)) = \sum_{i=1}^{r}(\frac{2\sqrt{m_v(vv^i)m_{v^i}(vv^i)}}{m_v(vv^i)+m_{v^i}(vv^i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_v(vv_i)m_{v_i}(vv_i)}}{m_v(vv_i)+m_{v_i}(vv_i)})^k + \sum_{i=1}^{n-1}(\frac{2\sqrt{m_{v_i}(v_iv_{i+1})m_{v_{i+1}}(v_iv_{i+1})}}{m_{v_i}(v_iv_{i+1})+m_{v_{i+1}}(v_iv_{i+1})})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{m_{v_i}(v_iv_i^j)m_{v_i^j}(v_iv_i^j)}}{m_{v_i}(v_iv_i^j)+m_{v_i^j}(v_iv_i^j)})^k$$

$$= r(\frac{2\sqrt{2n+r+nr-2}}{2n+r+nr-1})^k + (2(\frac{2\sqrt{(2n+nr-r-4)(r+1)}}{2n+nr-3})^k + 2(\frac{2\sqrt{(2n+nr-2r-4)(r+2)}}{2n+nr-r-2})^k + (n-4)(\frac{2\sqrt{(2n+nr-2r-5)(r+2)}}{2n+nr-r-3})^k) + (2(\frac{2\sqrt{(r+1)(2r+3)}}{3r+4})^k + 2(\frac{2\sqrt{(2r+2)(2r+3)}}{4r+5})^k + (n-5)(\frac{2\sqrt{(2r+3)(2r+3)}}{4r+6})^k) + nr(\frac{2\sqrt{2n+r+nr-2}}{2n+r+nr-1})^k.$$

□

Corollary 1. $GA_3^k(F_n) = 2(\frac{2\sqrt{2n-4}}{2n-3})^k + 2(\frac{\sqrt{(2n-4)(r+2)}}{n-1})^k + (n-4)(\frac{2\sqrt{2(2n-5)}}{2n-3})^k + 2(\frac{\sqrt{3}}{2})^k + 2(\frac{2\sqrt{6}}{5})^k + (n-5)$.

Theorem 2. $GA_3^k(I_r(W_n)) = n(\frac{2\sqrt{(r+2)(2n+nr-2r-5)}}{2n+nr-r-3})^k + n + r(n+1)(\frac{2\sqrt{2n+r+nr-1}}{2n+r+nr})^k$.

Proof. Let $C_n=v_1v_2\ldots v_n$ and v_i^1, v_i^2, …, v_i^r be the r hanging vertices of v_i ($1\le i\le n$). Let v be a vertex in W_n beside C_n, and v^1, v^2, …, v^r be the r hanging vertices of v.

In view of the definition of general third geometric-arithmetic index, we infer

$$GA_3^k(I_r(W_n)) = \sum_{i=1}^{r}(\frac{2\sqrt{m_v(vv^i)m_{v^i}(vv^i)}}{m_v(vv^i)+m_{v^i}(vv^i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_v(vv_i)m_{v_i}(vv_i)}}{m_v(vv_i)+m_{v_i}(vv_i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_{v_i}(v_iv_{i+1})m_{v_{i+1}}(v_iv_{i+1})}}{m_{v_i}(v_iv_{i+1})+m_{v_{i+1}}(v_iv_{i+1})})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{m_{v_i}(v_iv_i^j)m_{v_i^j}(v_iv_i^j)}}{m_{v_i}(v_iv_i^j)+m_{v_i^j}(v_iv_i^j)})^k$$

$$= r(\frac{2\sqrt{2n+r+nr-1}}{2n+r+nr})^k + n(\frac{2\sqrt{(r+2)(2n+nr-2r-5)}}{2n+nr-r-3})^k + n(\frac{2\sqrt{(2r+3)(2r+3)}}{4r+6})^k + nr(\frac{2\sqrt{2n+r+nr-1}}{2n+r+nr})^k.$$

□

Corollary 2. $GA_3^k(W_n) = n(\frac{2\sqrt{2(2n-5)}}{2n-3})^k + n$.

Theorem 3. $GA_3^k(I_r(\tilde{F}_n)) = 2(\frac{2\sqrt{(2r+1)(2nr+3n-2r-5)}}{2nr+3n-4})^k$

$+(3n-4)(\frac{2\sqrt{(3r+2)(2nr+3n-3r-7)}}{2nr+3n-5})^k$ +

$2nr(\frac{2\sqrt{3n+2nr-3}}{3n+2nr-2})^k$.

Proof. Let $P_n=v_1v_2\ldots v_n$ and $v_{i,i+1}$ be the adding vertex between v_i and v_{i+1}. Let v_i^1, v_i^2,…, v_i^r be the r hanging vertices of v_i ($1\le i\le n$). Let $v_{i,i+1}^1$, $v_{i,i+1}^2$,…, $v_{i,i+1}^r$ be the r hanging vertices of $v_{i,i+1}$ ($1\le i\le n$-1). Let v be a vertex in F_n beside P_n, and the r hanging vertices of v be v^1, v^2, …, v^r.

By virtue of the definition of general third geometric-arithmetic index, we yield

$$GA_3^k(I_r(\tilde{F}_n)) = \sum_{i=1}^{r}(\frac{2\sqrt{m_v(vv^i)m_{v^i}(vv^i)}}{m_v(vv^i)+m_{v^i}(vv^i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_v(vv_i)m_{v_i}(vv_i)}}{m_v(vv_i)+m_{v_i}(vv_i)})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{m_{v_i}(v_iv_i^j)m_{v_i^j}(v_iv_i^j)}}{m_{v_i}(v_iv_i^j)+m_{v_i^j}(v_iv_i^j)})^k + \sum_{i=1}^{n-1}(\frac{2\sqrt{m_{v_i}(v_iv_{i,i+1})m_{v_{i,i+1}}(v_iv_{i,i+1})}}{m_{v_i}(v_iv_{i,i+1})+m_{v_{i,i+1}}(v_iv_{i,i+1})})^k + \sum_{i=1}^{n-1}(\frac{2\sqrt{m_{v_{i,i+1}}(v_{i,i+1}v_{i+1})m_{v_{i+1}}(v_{i,i+1}v_{i+1})}}{m_{v_{i,i+1}}(v_{i,i+1}v_{i+1})+m_{v_{i+1}}(v_{i,i+1}v_{i+1})})^k + \sum_{i=1}^{n-1}\sum_{j=1}^{r}(\frac{2\sqrt{m_{v_{i,i+1}}(v_{i,i+1}v_{i,i+1}^j)m_{v_{i,i+1}^j}(v_{i,i+1}v_{i,i+1}^j)}}{m_{v_{i,i+1}}(v_{i,i+1}v_{i,i+1}^j)+m_{v_{i,i+1}^j}(v_{i,i+1}v_{i,i+1}^j)})^k$$

$$= r(\frac{2\sqrt{3n+2nr-3}}{3n+2nr-2})^k + (2(\frac{2\sqrt{(2r+1)(2nr+3n-2r-5)}}{2nr+3n-4})^k + (n-2)(\frac{2\sqrt{(3r+2)(2nr+3n-3r-7)}}{2nr+3n-5})^k) + nr(\frac{2\sqrt{3n+2nr-3}}{3n+2nr-2})^k + (n-1)(\frac{2\sqrt{(3r+2)(2nr-3r+3n-7)}}{2nr+3n-5})^k + (n-1)(\frac{2\sqrt{(3r+2)(2nr-3r+3n-7)}}{2nr+3n-5})^k + (n-1)r(\frac{2\sqrt{3n+2nr-3}}{3n+2nr-2})^k.$$

□

Corollary 3. $GA_3^k(\tilde{F}_n) = 2(\frac{2\sqrt{3n-5}}{3n-4})^k + (3n-4)(\frac{2\sqrt{2(3n-7)}}{3n-5})^k$.

Theorem 4. $GA_3^k(I_r(\tilde{W}_n)) = 3n(\frac{2\sqrt{(3r+2)(2nr+3n-2r-5)}}{2nr+3n+r-3})^k + r(2n+1)(\frac{2\sqrt{2nr+3n+r-1}}{2nr+3n+r})^k$.

Proof. Let $C_n=v_1v_2\ldots v_n$ and v be a vertex in W_n beside C_n, $v_{i,i+1}$ be the adding vertex between v_i and v_{i+1}. Let v^1, v^2, …, v^r be the r hanging vertices of v and v_i^1, v_i^2,…, v_i^r be the r hanging vertices of v_i ($1\le i\le n$). Let $v_{n,n+1}=v_{1,n}$ and $v_{i,i+1}^1$, $v_{i,i+1}^2$,…, $v_{i,i+1}^r$ be the r hanging vertices of $v_{i,i+1}$ ($1\le i\le n$).

In view of the definition of general third geometric-arithmetic index, we deduce

$$GA_3^k(I_r(\tilde{W}_n)) = \sum_{i=1}^{r}(\frac{2\sqrt{m_v(vv^i)m_{v^i}(vv^i)}}{m_v(vv^i)+m_{v^i}(vv^i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_v(vv_i)m_{v_i}(vv_i)}}{m_v(vv_i)+m_{v_i}(vv_i)})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{m_{v_i}(v_iv_i^j)m_{v_i^j}(v_iv_i^j)}}{m_{v_i}(v_iv_i^j)+m_{v_i^j}(v_iv_i^j)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_{v_i}(v_iv_{i,i+1})m_{v_{i,i+1}}(v_iv_{i,i+1})}}{m_{v_i}(v_iv_{i,i+1})+m_{v_{i,i+1}}(v_iv_{i,i+1})})^k + \sum_{i=1}^{n}(\frac{2\sqrt{m_{v_{i,i+1}}(v_{i,i+1}v_{i+1})m_{v_{i+1}}(v_{i,i+1}v_{i+1})}}{m_{v_{i,i+1}}(v_{i,i+1}v_{i+1})+m_{v_{i+1}}(v_{i,i+1}v_{i+1})})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{m_{v_{i,i+1}}(v_{i,i+1}v_{i,i+1}^j)m_{v_{i,i+1}^j}(v_{i,i+1}v_{i,i+1}^j)}}{m_{v_{i,i+1}}(v_{i,i+1}v_{i,i+1}^j)+m_{v_{i,i+1}^j}(v_{i,i+1}v_{i,i+1}^j)})^k$$

$$= r(\frac{2\sqrt{2nr+3n+r-1}}{2nr+3n+r})^k + n(\frac{2\sqrt{(3r+2)(2nr+3n-2r-5)}}{2nr+3n+r-3})^k + nr(\frac{2\sqrt{2nr+3n+r-1}}{2nr+3n+r})^k + n(\frac{2\sqrt{(3r+2)(2nr+3n-2r-5)}}{2nr+3n+r-3})^k + n(\frac{2\sqrt{(3r+2)(2nr+3n-2r-5)}}{2nr+3n+r-3})^k$$

$+ nr(\frac{2\sqrt{2nr+3n+r-1}}{2nr+3n+r})^k$.

□

Corollary 4. $GA_3^k(\tilde{W}_n) = 3n(\frac{2\sqrt{2(3n-5)}}{3n-3})^k$.

3. Generalized Zagreb Index

Theorem 5. $M_{\{r,s\}}(I_r(F_n)) = (r^2-r) + 2r((r+n)^{t_1}+(r+n)^{t_2}) + 2((n+r)^{t_1}(2+r)^{t_2}+(n+r)^{t_2}(2+r)^{t_1}) + (n-2)((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1}) + 2nr((r+n)^{t_1}+(r+n)^{t_2}) + 2r((r+2)^{t_1}+(r+2)^{t_2}) + (n-2)r((r+3)^{t_1}+(r+3)^{t_2}) + 2nr^2 + 2(r+2)^{t_1+t_2} + 2(n-2)((r+2)^{t_1}(r+3)^{t_2} + (r+2)^{t_2}(r+3)^{t_1}) + (n-2)(n-3)(r+3)^{t_1+t_2} + 2r((r+2)^{t_1}+(r+2)^{t_2}) + (n-2)r((r+3)^{t_1}+(r+3)^{t_2}) + 2(n-1)r((r+2)^{t_1} + (r+2)^{t_2}) + (n-2)(n-1)r((r+3)^{t_1}+(r+3)^{t_2}) + nr(r-1) + 2r^2n(n-1)$.

Proof. By the definition of generalized Zagreb index, we have

$$M_{\{r,s\}}(I_r(F_n)) = \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(d(v^i)^{t_1}d(v^j)^{t_2}+d(v^i)^{t_2}d(v^j)^{t_1}) + \sum_{i=1}^{r}(d(v)^{t_1}d(v^i)^{t_2}+d(v)^{t_2}d(v^i)^{t_1}) + \sum_{i=1}^{n}(d(v)^{t_1}d(v_i)^{t_2}+d(v)^{t_2}d(v_i)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}(d(v)^{t_1}d(v_i^j)^{t_2}+d(v)^{t_2}d(v_i^j)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v^j)^{t_2}+d(v_i)^{t_2}d(v^j)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{r}(d(v_i^j)^{t_1}d(v^k)^{t_2}+d(v_i^j)^{t_2}d(v^k)^{t_1}) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d(v_i)^{t_1}d(v_j)^{t_2}+d(v_i)^{t_2}d(v_j)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v_i^j)^{t_2}+d(v_i)^{t_2}d(v_i^j)^{t_1}) + \sum_{i=1}^{n}\sum_{j\in\{1,2,\cdots,n\}-i}\sum_{k=1}^{r}(d(v_i)^{t_1}d(v_j^k)^{t_2}+d(v_i)^{t_2}d(v_j^k)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}(d(v_i^j)^{t_1}d(v_i^k)^{t_2}+d(v_i^j)^{t_2}d(v_i^k)^{t_1}) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}(d(v_i^k)^{t_1}d(v_j^t)^{t_2}+d(v_i^k)^{t_2}d(v_j^t)^{t_1})$$

$$= (r^2-r) + 2r((r+n)^{t_1}+(r+n)^{t_2}) + 2((n+r)^{t_1}(2+r)^{t_2}+(n+r)^{t_2}(2+r)^{t_1}) + (n-2)((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1}) + 2nr((r+n)^{t_1}+(r+n)^{t_2}) + 2r((r+2)^{t_1}+(r+2)^{t_2}) + (n-2)r((r+3)^{t_1}+(r+3)^{t_2}) + 2nr^2 + 2(r+2)^{t_1+t_2} + 2(n-2)((r+2)^{t_1}(r+3)^{t_2}+(r+2)^{t_2}(r+3)^{t_1}) + (n-2)(n-3)(r+3)^{t_1+t_2} + 2r((r+2)^{t_1}+(r+2)^{t_2}) + (n-2)r((r+3)^{t_1}+(r+3)^{t_2}) + 2(n-1)r((r+2)^{t_1} + (r+2)^{t_2}) + (n-2)(n-1)r((r+3)^{t_1}+(r+3)^{t_2}) + nr(r-1) + 2r^2n(n-1).$$ □

Corollary 5. $M_{\{r,s\}}(F_n) = 2(n^{t_1}2^{t_2}+n^{t_2}2^{t_1}) + (n-2)(n^{t_1}3^{t_2}+n^{t_2}3^{t_1}) + 2\cdot 2^{t_1+t_2} + 2(n-2)(2^{t_1}3^{t_2}+2^{t_2}3^{t_1}) + (n-2)(n-3)3^{t_1+t_2}$.

Theorem 6. $M_{\{r,s\}}(I_r(W_n)) = (r^2-r) + 2r((r+n)^{t_1}+(r+n)^{t_2}) + n((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1}) + 2nr((r+n)^{t_1}+(r+n)^{t_2}) + nr((r+3)^{t_1}+(r+3)^{t_2}) + 2nr^2 + n(n-1)(r+3)^{t_1+t_2} + nr((r+3)^{t_1}+(r+3)^{t_2}) + (n-1)nr((r+3)^{t_1}+(r+3)^{t_2}) + nr(r-1) + 2r^2n(n-1)$.

Proof. By the definition of generalized Zagreb index, we have

$$M_{\{r,s\}}(I_r(W_n)) = \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(d(v^i)^{t_1}d(v^j)^{t_2}+d(v^i)^{t_2}d(v^j)^{t_1}) + \sum_{i=1}^{r}(d(v)^{t_1}d(v^i)^{t_2}+d(v)^{t_2}d(v^i)^{t_1}) + \sum_{i=1}^{n}(d(v)^{t_1}d(v_i)^{t_2}+d(v)^{t_2}d(v_i)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}(d(v)^{t_1}d(v_i^j)^{t_2}+d(v)^{t_2}d(v_i^j)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v^j)^{t_2}+d(v_i)^{t_2}d(v^j)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{r}(d(v_i^j)^{t_1}d(v^k)^{t_2}+d(v_i^j)^{t_2}d(v^k)^{t_1}) + \sum_{i=1}^{n}\sum_{j=i+1}^{n}(d(v_i)^{t_1}d(v_j)^{t_2}+d(v_i)^{t_2}d(v_j)^{t_1}) + \sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v_i^j)^{t_2}+d(v_i)^{t_2}d(v_i^j)^{t_1}) + \sum_{i=1}^{n}\sum_{j\in\{1,2,\cdots,n\}-i}\sum_{k=1}^{r}(d(v_i)^{t_1}d(v_j^k)^{t_2}+d(v_i)^{t_2}d(v_j^k)^{t_1}) +$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}(d(v_i^j)^{t_1}d(v_i^k)^{t_2}+d(v_i^j)^{t_2}d(v_i^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}(d(v_i^k)^{t_1}d(v_j^t)^{t_2}+d(v_i^k)^{t_2}d(v_j^t)^{t_1})$$

$$=(r^2-r)+2r((r+n)^{t_1}+(r+n)^{t_2})+$$
$$n((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1})+$$
$$2nr((r+n)^{t_1}+(r+n)^{t_2})+nr((r+3)^{t_1}+(r+3)^{t_2})$$
$$+2nr^2+n(n-1)(r+3)^{t_1+t_2}+nr((r+3)^{t_1}+(r+3)^{t_2})+$$
$$(n-1)nr((r+3)^{t_1}+(r+3)^{t_2})+nr(r-1)+2r^2n(n-1).$$

□

Corollary 6. $M_{\{r,s\}}(W_n)=n(n^{t_1}3^{t_2}+n^{t_2}3^{t_1})+n(n-1)3^{t_1+t_2}$.

Theorem 7. $M_{\{r,s\}}(I_r(\tilde{F}_n))=(r^2-r)+$

$$2r((r+n)^{t_1}+(r+n)^{t_2})+2((n+r)^{t_1}(2+r)^{t_2}$$
$$+(n+r)^{t_2}(2+r)^{t_1})+(n-2)((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1})$$
$$+2nr((r+n)^{t_1}+(r+n)^{t_2})+$$
$$2r((r+2)^{t_1}+(r+2)^{t_2})+(n-2)r((r+3)^{t_1}+(r+3)^{t_2})$$
$$+2nr^2+2r((r+2)^{t_1}+(r+2)^{t_2})$$
$$+(n-2)r((r+3)^{t_1}+(r+3)^{t_2})+$$
$$2(n-1)r((r+2)^{t_1}+(r+2)^{t_2})+(n-2)(n-1)r((r+3)^{t_1}+(r+3)^{t_2})$$
$$+nr(r-1)+2r^2n(n-1)+$$
$$(n-1)((n+r)^{t_1}(r+2)^{t_2}+(n+r)^{t_1}(r+2)^{t_2})+$$
$$(n-1)r((n+r)^{t_1}+(n+r)^{t_2})+$$
$$r(n-1)((2+r)^{t_1}+(2+r)^{t_2})+2(n-1)r^2+$$
$$2(n-1)((r+2)^{t_1}+(r+2)^{t_2})+(n-2)(n-1)((r+3)^{t_1}+(r+3)^{t_2})+$$
$$2(n-1)((r+2)^{t_1}+(r+2)^{t_2})+(n-2)(n-1)((r+3)^{t_1}+(r+3)^{t_2})$$
$$+n(n-1)r((r+2)^{t_1}+(r+2)^{t_2})+2n(n-1)r^2+$$
$$2(n-1)(n-2)(2+r)^{t_1+t_2}+(n-1)r((2+r)^{t_1}+(2+r)^{t_2})+$$
$$(n-1)(n-2)r((r+2)^{t_1}+(r+2)^{t_2})+2(n-1)r^2+$$
$$2(n-2)(n-1)r^2.$$

Proof. By virtue of the definition of generalized Zagreb index, we get

$$M_{\{r,s\}}(I_r(\tilde{F}_n))=$$

$$\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(d(v^i)^{t_1}d(v^j)^{t_2}+d(v^i)^{t_2}d(v^j)^{t_1})+$$

$$\sum_{i=1}^{r}(d(v)^{t_1}d(v^i)^{t_2}+d(v)^{t_2}d(v^i)^{t_1})+$$

$$\sum_{i=1}^{n}(d(v)^{t_1}d(v_i)^{t_2}+d(v)^{t_2}d(v_i)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v)^{t_1}d(v_i^j)^{t_2}+d(v)^{t_2}d(v_i^j)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v^j)^{t_2}+d(v_i)^{t_2}d(v^j)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{r}(d(v_i^j)^{t_1}d(v^k)^{t_2}+d(v_i^j)^{t_2}d(v^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v_i^j)^{t_2}+d(v_i)^{t_2}d(v_i^j)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j\in\{1,2,\cdots,n\}-i}\sum_{k=1}^{r}(d(v_i)^{t_1}d(v_j^k)^{t_2}+d(v_i)^{t_2}d(v_j^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}(d(v_i^j)^{t_1}d(v_i^k)^{t_2}+d(v_i^j)^{t_2}d(v_i^k)^{t_1})+$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}(d(v_i^k)^{t_1}d(v_j^t)^{t_2}+d(v_i^k)^{t_2}d(v_j^t)^{t_1})+$$

$$\sum_{i=1}^{n-1}(d(v)^{t_1}d(v_{i,i+1})^{t_2}+d(v)^{t_2}d(v_{i,i+1})^{t_1})+$$

$$\sum_{i=1}^{n-1}\sum_{j=1}^{r}(d(v)^{t_1}d(v^j_{i,i+1})^{t_2}+d(v)^{t_2}d(v^j_{i,i+1})^{t_1})+$$

$$\sum_{i=1}^{r}\sum_{j=1}^{n-1}(d(v^i)^{t_1}d(v_{j,j+1})^{t_2}+d(v^i)^{t_2}d(v_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}(d(v^i)^{t_1}d(v^k_{j,j+1})^{t_2}+d(v^i)^{t_2}d(v^k_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}(d(v_i)^{t_1}d(v_{j,j+1})^{t_2}+d(v_i)^{t_2}d(v_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}(d(v_i)^{t_1}d(v^k_{j,j+1})^{t_2}+d(v_i)^{t_2}d(v^k_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{n-1}(d(v_i^j)^{t_1}d(v_{k,k+1})^{t_2}+d(v_i^j)^{t_2}d(v_{k,k+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{t=1}^{r}(d(v_i^j)^{t_1}d(v^r_{k,k+1})^{t_2}+d(v_i^j)^{t_2}d(v^r_{k,k+1})^{t_1})+$$

$$\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}(d(v_{i,i+1})^{t_1}d(v_{j,j+1})^{t_2}+d(v_{i,i+1})^{t_2}d(v_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{n-1}\sum_{j=1}^{r}(d(v_{i,i+1})^{t_1}d(v^j_{i,i+1})^{t_2}+d(v_{i,i+1})^{t_2}d(v^j_{i,i+1})^{t_1})+$$

$$\sum_{i=1}^{n-1}\sum_{j\in\{1,2,\cdots,n-1\}-i}\sum_{k=1}^{r}(d(v_{i,i+1})^{t_1}d(v^k_{j,j+1})^{t_2}+d(v_{i,i+1})^{t_2}d(v^k_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{n-1}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}(d(v^j_{i,i+1})^{t_1}d(v^k_{i,i+1})^{t_2}+d(v^j_{i,i+1})^{t_2}d(v^k_{i,i+1})^{t_1})+$$

$$\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=1}^{r}\sum_{t=1}^{r}(d(v^k_{i,i+1})^{t_1}d(v^t_{j,j+1})^{t_2}+d(v^k_{i,i+1})^{t_2}d(v^t_{j,j+1})^{t_1})$$

$$=(r^2-r)+2r((r+n)^{t_1}+(r+n)^{t_2})+$$
$$2((n+r)^{t_1}(2+r)^{t_2}+(n+r)^{t_2}(2+r)^{t_1})+(n-2)((n+r)^{t_1}(3+r)^{t_2}$$

$+(n+r)^{t_2}(3+r)^{t_1})+2nr((r+n)^{t_1}+(r+n)^{t_2})+$

$2r((r+2)^{t_1}+(r+2)^{t_2})+(n-2)r((r+3)^{t_1}+(r+3)^{t_2})+2nr^2+$

$2r((r+2)^{t_1}+(r+2)^{t_2})+(n-2)r((r+3)^{t_1}+(r+3)^{t_2})+$

$2(n-1)r((r+2)^{t_1}+(r+2)^{t_2})$

$+(n-2)(n-1)r((r+3)^{t_1}+(r+3)^{t_2})+nr(r-1)+2r^2n(n-1)+$

$(n-1)((n+r)^{t_1}(r+2)^{t_2}+(n+r)^{t_1}(r+2)^{t_2})+$

$(n-1)r((n+r)^{t_1}+(n+r)^{t_2})+$

$r(n-1)((2+r)^{t_1}+(2+r)^{t_2})+2(n-1)r^2+$

$2(n-1)((r+2)^{t_1}+(r+2)^{t_2})$

$+(n-2)(n-1)((r+3)^{t_1}+(r+3)^{t_2})+$

$2(n-1)((r+2)^{t_1}+(r+2)^{t_2})+(n-2)(n-1)((r+3)^{t_1}+(r+3)^{t_2})$

$+n(n-1)r((r+2)^{t_1}+(r+2)^{t_2})+2n(n-1)r^2+$

$2(n-1)(n-2)(2+r)^{t_1+t_2}+(n-1)r((2+r)^{t_1}+(2+r)^{t_2})+$

$(n-1)(n-2)r((r+2)^{t_1}+(r+2)^{t_2})+2(n-1)r^2+$

$2(n-2)(n-1)r^2$.. □

Corollary 7. $M_{\{r,s\}}(\tilde{F}_n)=$

$2(n^{t_1}2^{t_2}+n^{t_2}2^{t_1})+(n-2)(n^{t_1}3^{t_2}+n^{t_2}3^{t_1})+$

$(n-1)(n^{t_1}2^{t_2}+n^{t_1}2^{t_2})+$

$2(n-1)(2^{t_1}+2^{t_2})+(n-2)(n-1)(3^{t_1}+3^{t_2})+$

$2(n-1)(2^{t_1}+2^{t_2})+(n-2)(n-1)(3^{t_1}+3^{t_2})+$

$2(n-1)(n-2)2^{t_1+t_2}$.

Theorem 8. $M_{\{r,s\}}(I_r(\tilde{W}_n))=(r^2-r)+$

$2r((r+n)^{t_1}+(r+n)^{t_2})+n((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1})$

$+2nr((r+n)^{t_1}+(r+n)^{t_2})+nr((r+3)^{t_1}+(r+3)^{t_2})+2nr^2+$

$nr((r+3)^{t_1}+(r+3)^{t_2})+n(n-1)r((r+3)^{t_1}+(r+3)^{t_2})+nr(r-1)+2r^2n(n-1)+$

$(n-1)((n+r)^{t_1}(r+2)^{t_2}+(n+r)^{t_1}(r+2)^{t_2})+$

$(n-1)r((n+r)^{t_1}+(n+r)^{t_2})+$

$r(n-1)((2+r)^{t_1}+(2+r)^{t_2})+2(n-1)r^2+$

$n(n-1)((r+3)^{t_1}+(r+3)^{t_2})+$

$n(n-1)((r+3)^{t_1}+(r+3)^{t_2})+$

$n(n-1)r((r+2)^{t_1}+(r+2)^{t_2})+2n(n-1)r^2+$

$2(n-1)(n-2)(2+r)^{t_1+t_2}+(n-1)r((2+r)^{t_1}+(2+r)^{t_2})+$

$(n-1)(n-2)r((r+2)^{t_1}+(r+2)^{t_2})+2(n-1)r^2+$

$2(n-2)(n-1)r^2$.

Proof. In view of the definition of generalized Zagreb index, we deduce

$M_{\{r,s\}}(I_r(\tilde{W}_n))=$

$$\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(d(v^i)^{t_1}d(v^j)^{t_2}+d(v^i)^{t_2}d(v^j)^{t_1})+$$

$$\sum_{i=1}^{r}(d(v)^{t_1}d(v^i)^{t_2}+d(v)^{t_2}d(v^i)^{t_1})+$$

$$\sum_{i=1}^{n}(d(v)^{t_1}d(v_i)^{t_2}+d(v)^{t_2}d(v_i)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v)^{t_1}d(v_i^j)^{t_2}+d(v)^{t_2}d(v_i^j)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v^j)^{t_2}+d(v_i)^{t_2}d(v^j)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{r}(d(v_i^j)^{t_1}d(v^k)^{t_2}+d(v_i^j)^{t_2}d(v^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_i)^{t_1}d(v_i^j)^{t_2}+d(v_i)^{t_2}d(v_i^j)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j\in\{1,2,\cdots,n\}-i}\sum_{k=1}^{r}(d(v_i)^{t_1}d(v_j^k)^{t_2}+d(v_i)^{t_2}d(v_j^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}(d(v_i^j)^{t_1}d(v_i^k)^{t_2}+d(v_i^j)^{t_2}d(v_i^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}(d(v_i^k)^{t_1}d(v_j^t)^{t_2}+d(v_i^k)^{t_2}d(v_j^t)^{t_1})+$$

$$\sum_{i=1}^{n}(d(v)^{t_1}d(v_{i,i+1})^{t_2}+d(v)^{t_2}d(v_{i,i+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v)^{t_1}d(v_{i,i+1}^j)^{t_2}+d(v)^{t_2}d(v_{i,i+1}^j)^{t_1})+$$

$$\sum_{i=1}^{r}\sum_{j=1}^{n-1}(d(v^i)^{t_1}d(v_{j,j+1})^{t_2}+d(v^i)^{t_2}d(v_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}(d(v^i)^{t_1}d(v_{j,j+1}^k)^{t_2}+d(v^i)^{t_2}d(v_{j,j+1}^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}(d(v_i)^{t_1}d(v_{j,j+1})^{t_2}+d(v_i)^{t_2}d(v_{j,j+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}(d(v_i)^{t_1}d(v_{j,j+1}^k)^{t_2}+d(v_i)^{t_2}d(v_{j,j+1}^k)^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{n-1}(d(v_i^j)^{t_1}d(v_{k,k+1})^{t_2}+d(v_i^j)^{t_2}d(v_{k,k+1})^{t_1})+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{t=1}^{r}(d(v_i^j)^{t_1}d(v_{k,k+1}^r)^{t_2}+d(v_i^j)^{t_2}d(v_{k,k+1}^r)^{t_1})+$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d(v_{i,i+1})^{t_1}d(v_{j,j+1})^{t_2}+d(v_{i,i+1})^{t_2}d(v_{j,j+1})^{t_1})+$$

$\sum_{i=1}^{n}\sum_{j=1}^{r}(d(v_{i,i+1})^{t_1}d(v_{i,i+1}^{j})^{t_2}+d(v_{i,i+1})^{t_2}d(v_{i,i+1}^{j})^{t_1})$ +

$\sum_{i=1}^{n}\sum_{j\in\{1,2,\cdots,n\}-i}\sum_{k=1}^{r}(d(v_{i,i+1})^{t_1}d(v_{j,j+1}^{k})^{t_2}+d(v_{i,i+1})^{t_2}d(v_{j,j+1}^{k})^{t_1})$ +

$\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}(d(v_{i,i+1}^{j})^{t_1}d(v_{i,i+1}^{k})^{t_2}+d(v_{i,i+1}^{j})^{t_2}d(v_{i,i+1}^{k})^{t_1})$ +

$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-2}\sum_{k=1}^{r}\sum_{t=1}^{r}(d(v_{i,i+1}^{k})^{t_1}d(v_{j,j+1}^{t})^{t_2}+d(v_{i,i+1}^{k})^{t_2}d(v_{j,j+1}^{t})^{t_1})$

$=(r^2-r)+2r((r+n)^{t_1}+(r+n)^{t_2})+n((n+r)^{t_1}(3+r)^{t_2}+(n+r)^{t_2}(3+r)^{t_1})+2nr((r+n)^{t_1}+(r+n)^{t_2})+nr((r+3)^{t_1}+(r+3)^{t_2})+2nr^2+nr((r+3)^{t_1}+(r+3)^{t_2})+n(n-1)r((r+3)^{t_1}+(r+3)^{t_2})+nr(r-1)+2r^2n(n-1)+(n-1)((n+r)^{t_1}(r+2)^{t_2}+(n+r)^{t_1}(r+2)^{t_2})+(n-1)r((n+r)^{t_1}+(n+r)^{t_2})+r(n-1)((2+r)^{t_1}+(2+r)^{t_2})+2(n-1)r^2+n(n-1)((r+3)^{t_1}+(r+3)^{t_2})+n(n-1)((r+3)^{t_1}+(r+3)^{t_2})+n(n-1)r((r+2)^{t_1}+(r+2)^{t_2})+2n(n-1)r^2+2(n-1)(n-2)(2+r)^{t_1+t_2}+(n-1)r((2+r)^{t_1}+(2+r)^{t_2})+(n-1)(n-2)r((r+2)^{t_1}+(r+2)^{t_2})+2(n-1)r^2+2(n-2)(n-1)r^2$. □

Corollary 8. $M_{\{r,s\}}(\tilde{W}_n)=n(n^{t_1}3^{t_2}+n^{t_2}3^{t_1})+(n-1)(n^{t_1}2^{t_2}+n^{t_1}2^{t_2})+n(n-1)(3^{t_1}+3^{t_2})+n(n-1)(3^{t_1}+3^{t_2})+2(n-1)(n-2)2^{t_1+t_2}$.

4. **General Harmonic Index**

Theorem 9. $H_k(I_r(F_n))=r(\frac{2}{n+r+1})^k+2(\frac{2}{n+2r+2})^k+(n-2)(\frac{2}{n+2r+3})^k+2(\frac{2}{2r+5})^k+(n-3)(\frac{1}{r+3})^k+2r(\frac{2}{r+3})^k+(n-2)r(\frac{2}{r+4})^k$.

Proof. In view of the definition of general harmonic index, we deduce

$H_k(I_r(F_n))=\sum_{i=1}^{r}(\frac{2}{d(v)+d(v^i)})^k+\sum_{i=1}^{n}(\frac{2}{d(v)+d(v_i)})^k+\sum_{i=1}^{n-1}(\frac{2}{d(v_i)+d(v_{i+1})})^k+\sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2}{d(v_i)+d(v_i^j)})^k$

$=r(\frac{2}{n+r+1})^k+(2(\frac{2}{n+2r+2})^k+(n-2)(\frac{2}{n+2r+3})^k)+(2(\frac{2}{2r+5})^k+(n-3)(\frac{2}{2r+6})^k)+(2r(\frac{2}{r+3})^k+(n-2)r(\frac{2}{r+4})^k)$.

□

Corollary 9. $H_k(F_n)=2(\frac{2}{n+2})^k+(n-2)(\frac{2}{n+3})^k+2(\frac{2}{5})^k+(n-3)(\frac{1}{3})^k$.

Theorem 10. $H_k(I_r(W_n))=r(\frac{2}{n+r+1})^k+n(\frac{2}{n+2r+3})^k+n(\frac{1}{r+3})^k+nr(\frac{2}{r+4})^k$.

Proof. In terms of the definition of general harmonic index, we infer

$H_k(I_r(W_n))=\sum_{i=1}^{r}(\frac{2}{d(v)+d(v^i)})^k+\sum_{i=1}^{n}(\frac{2}{d(v)+d(v_i)})^k+\sum_{i=1}^{n}(\frac{2}{d(v_i)+d(v_{i+1})})^k+\sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2}{d(v_i)+d(v_i^j)})^k$

$=r(\frac{2}{n+r+1})^k+n(\frac{2}{n+2r+3})^k+n(\frac{2}{2r+6})^k+nr(\frac{2}{r+4})^k$. □

Corollary 10. $H_k(W_n)=n(\frac{2}{n+3})^k+n(\frac{1}{3})^k$.

Theorem 11. $H_k(I_r(\tilde{F}_n))=r(\frac{2}{n+r+1})^k+2(\frac{2}{n+2r+2})^k+(n-2)(\frac{2}{n+2r+3})^k+(n-2)r(\frac{2}{r+4})^k+2(\frac{1}{r+2})^k+2(n-2)(\frac{2}{2r+5})^k+(n+1)r(\frac{2}{r+3})^k$.

Proof. Using the definition of general harmonic index, we obtain

$H_k(I_r(\tilde{F}_n))=\sum_{i=1}^{r}(\frac{2}{d(v)+d(v^i)})^k+\sum_{i=1}^{n}(\frac{2}{d(v)+d(v_i)})^k+\sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2}{d(v_i)+d(v_i^j)})^k+$

$$\sum_{i=1}^{n-1}\left(\frac{2}{d(v_i)+d(v_{i,i+1})}\right)^k + \sum_{i=1}^{n-1}\left(\frac{2}{d(v_{i,i+1})+d(v_{i+1})}\right)^k + \sum_{i=1}^{n-1}\sum_{j=1}^{r}\left(\frac{2}{d(v_{i,i+1})+d(v^j_{i,i+1})}\right)^k$$

$$= r\left(\frac{2}{n+r+1}\right)^k + \left(2\left(\frac{2}{n+2r+2}\right)^k+(n-2)\left(\frac{2}{n+2r+3}\right)^k\right) + \left(2r\left(\frac{2}{r+3}\right)^k+(n-2)r\left(\frac{2}{r+4}\right)^k\right) + \left(\left(\frac{2}{2r+4}\right)^k+(n-2)\left(\frac{2}{2r+5}\right)^k\right) + \left(\left(\frac{2}{2r+4}\right)^k+(n-2)\left(\frac{2}{2r+5}\right)^k\right) + (n-1)r\left(\frac{2}{r+3}\right)^k.$$

□

Corollary 11. $H_k(\tilde{F}_n) = 2\left(\frac{2}{n+2}\right)^k+(n-2)\left(\frac{2}{n+3}\right)^k + 2\left(\frac{1}{2}\right)^k+2(n-2)\left(\frac{2}{5}\right)^k$.

Theorem 12. $H_k(I_r(\tilde{W}_n)) = r\left(\frac{2}{n+r+1}\right)^k + n\left(\frac{2}{n+2r+3}\right)^k + nr\left(\frac{2}{r+4}\right)^k + 2n\left(\frac{2}{2r+5}\right)^k + nr\left(\frac{2}{r+3}\right)^k$.

Proof. By virtue of the definition of general harmonic index, we get

$$H_k(I_r(\tilde{W}_n)) = \sum_{i=1}^{r}\left(\frac{2}{d(v)+d(v^i)}\right)^k + \sum_{i=1}^{n}\left(\frac{2}{d(v)+d(v_i)}\right)^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left(\frac{2}{d(v_i)+d(v_i^j)}\right)^k + \sum_{i=1}^{n}\left(\frac{2}{d(v_i)+d(v_{i,i+1})}\right)^k + \sum_{i=1}^{n}\left(\frac{2}{d(v_{i,i+1})+d(v_{i+1})}\right)^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left(\frac{2}{d(v_{i,i+1})+d(v^j_{i,i+1})}\right)^k$$

$$= r\left(\frac{2}{n+r+1}\right)^k + n\left(\frac{2}{n+2r+3}\right)^k + nr\left(\frac{2}{r+4}\right)^k + n\left(\frac{2}{2r+5}\right)^k + n\left(\frac{2}{2r+5}\right)^k + nr\left(\frac{2}{r+3}\right)^k.$$ □

Corollary 12. $H_k(\tilde{W}_n) = n\left(\frac{2}{n+3}\right)^k + 2n\left(\frac{2}{5}\right)^k$.

5. General Second Geometric-arithmetic Index

Theorem 13. $GA_2^k(I_r(F_n)) = 2\left(\frac{2\sqrt{n-1}}{n}\right)^k+(n-2)\left(\frac{2\sqrt{n-2}}{n-1}\right)^k + 2\left(\frac{2\sqrt{2}}{3}\right)^k + (n-3) + (n+1)r\left(\frac{2\sqrt{r+n(r+1)}}{(n+1)(r+1)}\right)^k$.

Proof. Using the definition of general second geometric-arithmetic index, we have

$$GA_2^k(I_r(F_n)) = \sum_{i=1}^{r}\left(\frac{2\sqrt{n(v)n(v^i)}}{n(v)+n(v^i)}\right)^k + \sum_{i=1}^{n}\left(\frac{2\sqrt{n(v)n(v_i)}}{n(v)+n(v_i)}\right)^k + \sum_{i=1}^{n-1}\left(\frac{2\sqrt{n(v_i)n(v_{i+1})}}{n(v_i)+n(v_{i+1})}\right)^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left(\frac{2\sqrt{n(v_i)n(v_i^j)}}{n(v_i)+n(v_i^j)}\right)^k$$

$$= r\left(\frac{2\sqrt{r+n(r+1)}}{(n+1)(r+1)}\right)^k + \left(2\left(\frac{2\sqrt{(r+1)[(n-1)(r+1)]}}{n(r+1)}\right)^k + (n-2)\left(\frac{2\sqrt{(r+1)[(n-2)(r+1)]}}{(n-1)(r+1)}\right)^k\right) + 2\left(\frac{2\sqrt{(r+1)2(r+1)}}{3(r+1)}\right)^k + (n-3)\left(\frac{2\sqrt{2(r+1)2(r+1)}}{4(r+1)}\right)^k + nr\left(\frac{2\sqrt{r+n(r+1)}}{(n+1)(r+1)}\right)^k.$$ □

Corollary 13. $GA_2^k(F_n) = 2\left(\frac{2\sqrt{n-1}}{n}\right)^k + (n-2)\left(\frac{2\sqrt{n-2}}{(n-1)}\right)^k + 2\left(\frac{2\sqrt{2}}{3}\right)^k + (n-3)$.

Theorem 14. $GA_2^k(I_r(W_n)) = n\left(\frac{2\sqrt{n-2}}{n-1}\right)^k + n + r(n+1)\left(\frac{2\sqrt{r+n(r+1)}}{(n+1)(r+1)}\right)^k$.

Proof. In view of the definition of general second geometric-arithmetic index, we infer

$$GA_2^k(I_r(W_n)) = \sum_{i=1}^{r}\left(\frac{2\sqrt{n(v)n(v^i)}}{n(v)+n(v^i)}\right)^k + \sum_{i=1}^{n}\left(\frac{2\sqrt{n(v)n(v_i)}}{n(v)+n(v_i)}\right)^k + \sum_{i=1}^{n}\left(\frac{2\sqrt{n(v_i)n(v_{i+1})}}{n(v_i)+n(v_{i+1})}\right)^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left(\frac{2\sqrt{n(v_i)n(v_i^j)}}{n(v_i)+n(v_i^j)}\right)^k$$

$$= r\left(\frac{2\sqrt{r+n(r+1)}}{(n+1)(r+1)}\right)^k + n\left(\frac{2\sqrt{(n-2)(r+1)(r+1)}}{(n-1)(r+1)}\right)^k + n\left(\frac{2\sqrt{2(1+r)2(1+r)}}{4(r+1)}\right)^k + nr\left(\frac{2\sqrt{r+n(r+1)}}{(n+1)(r+1)}\right)^k.$$ □

Corollary 14. $GA_2^k(W_n) = n\left(\frac{2\sqrt{n-2}}{n-1}\right)^k + n$.

Theorem 15. $GA_2^k(I_r(\tilde{F}_n)) = 2(\frac{2\sqrt{n-1}}{n})^k + 2nr(\frac{\sqrt{2n(r+1)-1}}{(r+1)n})^k + (3n-4)(\frac{\sqrt{3(2n-3)}}{n})^k$.

Proof. By virtue of the definition of general second geometric-arithmetic index, we yield

$$GA_2^k(I_r(\tilde{F}_n)) = \sum_{i=1}^{r}(\frac{2\sqrt{n(v)n(v^i)}}{n(v)+n(v^i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{n(v)n(v_i)}}{n(v)+n(v_i)})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{n(v_i)n(v_i^j)}}{n(v_i)+n(v_i^j)})^k + \sum_{i=1}^{n-1}(\frac{2\sqrt{n(v_i)n(v_{i,i+1})}}{n(v_i)+n(v_{i,i+1})})^k + \sum_{i=1}^{n-1}(\frac{2\sqrt{n(v_{i,i+1})n(v_{i+1})}}{n(v_{i,i+1})+n(v_{i+1})})^k + \sum_{i=1}^{n-1}\sum_{j=1}^{r}(\frac{2\sqrt{n(v_{i,i+1})n(v_{i,i+1}^j)}}{n(v_{i,i+1})+n(v_{i,i+1}^j)})^k$$

$$= r(\frac{2\sqrt{r+(r+1)(2n-1)}}{(r+1)2n})^k + 2(\frac{2\sqrt{(2n-2)(r+1)2(r+1)}}{2n(r+1)})^k + (n-2)(\frac{2\sqrt{(2n-3)(r+1)3(r+1)}}{2n(r+1)})^k + nr(\frac{2\sqrt{2n(r+1)-1}}{2n(r+1)})^k + (n-1)(\frac{2\sqrt{(2n-3)(r+1)3(r+1)}}{2n(r+1)})^k + (n-1)(\frac{2\sqrt{(2n-3)(r+1)3(r+1)}}{2n(r+1)})^k + (n-1)r(\frac{2\sqrt{2n(r+1)-1}}{2n(r+1)})^k.$$

□

Corollary 15. $GA_2^k(\tilde{F}_n) = 2(\frac{2\sqrt{n-1}}{n})^k + (3n-4)(\frac{\sqrt{3(2n-3)}}{n})^k$.

Theorem 16. $GA_2^k(I_r(\tilde{W}_n)) = 3n(\frac{2\sqrt{3(2n-2)}}{2n+1})^k + r(2n+1)(\frac{2\sqrt{r+2n(r+1)}}{(2n+1)(r+1)})^k$.

Proof. In view of the definition of general second geometric-arithmetic index, we deduce

$$GA_2^k(I_r(\tilde{W}_n)) = \sum_{i=1}^{r}(\frac{2\sqrt{n(v)n(v^i)}}{n(v)+n(v^i)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{n(v)n(v_i)}}{n(v)+n(v_i)})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{n(v_i)n(v_i^j)}}{n(v_i)+n(v_i^j)})^k + \sum_{i=1}^{n}(\frac{2\sqrt{n(v_i)n(v_{i,i+1})}}{n(v_i)+n(v_{i,i+1})})^k + \sum_{i=1}^{n}(\frac{2\sqrt{n(v_{i,i+1})n(v_{i+1})}}{n(v_{i,i+1})+n(v_{i+1})})^k + \sum_{i=1}^{n}\sum_{j=1}^{r}(\frac{2\sqrt{n(v_{i,i+1})n(v_{i,i+1}^j)}}{n(v_{i,i+1})+n(v_{i,i+1}^j)})^k$$

$$= r(\frac{2\sqrt{r+2n(r+1)}}{(2n+1)(r+1)})^k + n(\frac{2\sqrt{(2n-2)(r+1)3(r+1)}}{(2n+1)(r+1)})^k + nr(\frac{2\sqrt{(2n+1)(r+1)-1}}{(2n+1)(r+1)})^k + n(\frac{2\sqrt{(2n-2)(r+1)3(r+1)}}{(2n+1)(r+1)})^k + n(\frac{2\sqrt{(2n-2)(r+1)3(r+1)}}{(2n+1)(r+1)})^k + nr(\frac{2\sqrt{(2n+1)(r+1)-1}}{(2n+1)(r+1)})^k.$$ □

Corollary 16. $GA_2^k(\tilde{W}_n) = 3n(\frac{2\sqrt{3(2n-2)}}{2n+1})^k$.

6. General Co-PI Index

Theorem 17. Let $n \geq 3$, then $Co-PI_v^k(I_r(F_n)) = 2((n-2)(r+1))^k + (n-2)((n-3)(r+1))^k + 2(r+1)^k + (n+1)r((r+n(r+1))-1)^k$.

Proof. Using the definition of general Co-PI index, we have

$$Co-PI_v^k(I_r(F_n)) = \sum_{i=1}^{r}|n(v)-n(v^i)|^k + \sum_{i=1}^{n}|n(v)-n(v_i)|^k + \sum_{i=1}^{n-1}|n(v_i)-n(v_{i+1})|^k + \sum_{i=1}^{n}\sum_{j=1}^{r}|n(v_i)-n(v_i^j)|^k$$

$$= r|(r+n(r+1))-1|^k + (2|(n-1)(r+1)-(r+1)|^k + (n-2)|(n-2)(r+1)-(r+1)|^k) + 2|2(r+1)-(r+1)|^k + (n-3)|2(r+1)-2(r+1)|^k + nr|(r+n(r+1))-1|^k.$$ □

Corollary 17. Let $n \geq 3$, then $Co-PI_v^k(F_n) = 2(n-2)^k + (n-2)(n-3)^k + 2$.

Theorem 18. Let $n \geq 3$, then $Co-PI_v^k(I_r(W_n)) = n((n-3)(r+1))^k + (n+1)r((r+n(r+1))-1)^k$.

Proof. In view of the definition of general Co-PI index, we infer

$$Co-PI_v^k(I_r(W_n)) = \sum_{i=1}^{r}|n(v)-n(v^i)|^k + \sum_{i=1}^{n}|n(v)-n(v_i)|^k + \sum_{i=1}^{n}|n(v_i)-n(v_{i+1})|^k + \sum_{i=1}^{n}\sum_{j=1}^{r}|n(v_i)-n(v_i^j)|^k$$

$= r\left|(r+n(r+1))-1\right|^k + n\left|(n-2)(r+1)-(r+1)\right|^k +$
$n\left|2(1+r)-2(1+r)\right|^k + nr\left|(r+n(r+1))-1\right|^k$. □

Corollary 18. Let $n \geq 3$, then $Co-PI_v^k(W_n) = n(n-3)^k$.

Theorem 19. Let $n \geq 2$, then $Co-PI_v^k(I_r(\tilde{F}_n)) = 2(2(n-2)(r+1))^k + 2nr(2n(r+1)-2)^k + (3n-4)((6n-10)(r+1))^k$.

Proof. By virtue of the definition of general Co-PI index, we yield

$$Co-PI_v^k(I_r(\tilde{F}_n)) = \sum_{i=1}^{r}\left|n(v)-n(v^i)\right|^k + \sum_{i=1}^{n}\left|n(v)-n(v_i)\right|^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left|n(v_i)-n(v_i^j)\right|^k + \sum_{i=1}^{n-1}\left|n(v_i)-n(v_{i,i+1})\right|^k + \sum_{i=1}^{n-1}\left|n(v_{i,i+1})-n(v_{i+1})\right|^k + \sum_{i=1}^{n-1}\sum_{j=1}^{r}\left|n(v_{i,i+1})-n(v^j_{i,i+1})\right|^k$$

$$= r\left|r+(r+1)(2n-1)-1\right|^k + 2\left|2(n-1)(r+1)-2(r+1)\right|^k + (n-2)\left|3(2n-3)(r+1)-(r+1)\right|^k + nr\left|2n(r+1)-2\right|^k + (n-1)\left|3(2n-3)(r+1)-(r+1)\right|^k + (n-1)\left|3(2n-3)(r+1)-(r+1)\right|^k + (n-1)r\left|2n(r+1)-2\right|^k.$$

□

Corollary 19. Let $n \geq 2$, then $Co-PI_v^k(\tilde{F}_n) = 2(2n-4)^k + (3n-4)(6n-10)^k$.

Theorem 20. Let $n \geq 2$, then $Co-PI_v^k(I_r(\tilde{W}_n)) = 3n((6n-7)(r+1))^k + (2n+1)r(r+2n(r+1)-1)^k$.

Proof. In view of the definition of general Co-PI index, we deduce

$$Co-PI_v^k(I_r(\tilde{W}_n)) = \sum_{i=1}^{r}\left|n(v)-n(v^i)\right|^k + \sum_{i=1}^{n}\left|n(v)-n(v_i)\right|^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left|n(v_i)-n(v_i^j)\right|^k + \sum_{i=1}^{n}\left|n(v_i)-n(v_{i,i+1})\right|^k + \sum_{i=1}^{n}\left|n(v_{i,i+1})-n(v_{i+1})\right|^k + \sum_{i=1}^{n}\sum_{j=1}^{r}\left|n(v_{i,i+1})-n(v^j_{i,i+1})\right|^k$$

$$= r\left|r+2n(r+1)-1\right|^k + n\left|3(2n-2)(r+1)-(r+1)\right|^k + nr\left|r+2n(r+1)-1\right|^k + n\left|3(2n-2)(r+1)-(r+1)\right|^k + n\left|3(2n-2)(r+1)-(r+1)\right|^k + nr\left|(2n+1)(r+1)-2\right|^k.$$

□

Corollary 20. Let $n \geq 2$, then $Co-PI_v^k(\tilde{W}_n) = 3n(6n-7)^k$.

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