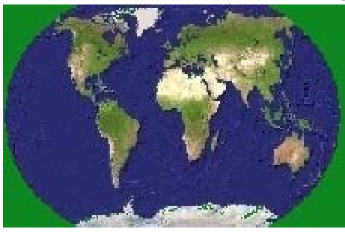
International Journal of Multidisciplinary Research and Development



International Journal of Multidisciplinary Research and Development

Volume :2, Issue :4, 273-277
April 2015
www.allsubjectjournal.com
e-ISSN: 2349-4182
p-ISSN: 2349-5979
Impact Factor: 3.762

Ahmed Mohammed Naji
Department of Studies in Mathematics University of Mysore, Manasagangotri
India

N. D. Soner
Department of Studies in Mathematics University of Mysore, Manasagangotri
India

Correspondence:
Ahmed Mohammed Naji
Department of Studies in Mathematics University of Mysore, Manasagangotri
India

The connected monopoly in graphs

Ahmed Mohammed Naji, N. D. Soner

Abstract
In a graph G(V, E), a set d $\subseteq$ V(G) is said to be a monopoly set of G if any vertex $v \in$V - D has at least $\frac{d(v)}{2}$ neighbors in *D*, where $d(v)$ is a degree of v in G. A monopoly set D of G is called a connected monopoly set of G if the subgraph $\langle D \rangle$ induced by D is connected. The minimum cardinalities of connected monopolies sets of G, denoted by $cmo(G)$ is called the connected monopoly size of G. In this paper, we investigate the relationship between $cmo(G)$ and some other parameters of graphs. Bounds for $cmo(G)$ and its exact values for some standard graphs are found.

Keywords: Connected monopoly set. Connected monopoly size.
Mathematics Subject Classification: 05C69, 05C99.

1. Introduction

In this paper, we are conceder with a simple graph $G(V,E)$, that nonempty, finite, have no loops no multiple, directed edges. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For subset $S \subseteq V(G)$, the subgraph $\langle S \rangle$ of G is called the subgraph induced by S if $E(\langle S \rangle) = \{uv \in E(G) \mid u, v \in S\}$. A graph G is said to be connected if for every pair of vertices there is a path joining them. The maximal connected subgraphs are called components. The connectivity number $\kappa(G)$ is defined as the minimum number of vertices whose removal from G results in a disconnected graph or in the trivial graph (a single vertex). A graph G is said to be k-connected if $\kappa(G) \geq k$. We refer to [4] for graph theory notation and terminology not described here.

A set D of vertices in a graph G is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. The concept of connected domination number was introduced by E. Sampathkumar and H. Walikar [8]. A dominating set D of a graph G is connected dominating set if a subgraph induced by D is connected. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set in G. for more details in domination theory of graphs we refer to [5].

A subset D of vertices set of a graph G is called a monopoly set if for every vertex $v \in V(G) - D$ has at least $\frac{d(v)}{2}$ neighbors in D. The monopoly size of G is the smallest cardinality of a monopoly set in G, denoted by $mo(G)$. A monopoly set D of a graph G is minimum if for any other monopoly set D' of G, $|D| \leq |D'|$. Any monopoly set D of a graph G with minimum cardinality is called a minimum monopoly set. In particular, monopolies are a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg [9]. For more details in dynamos in graphs we refer to [1, 2, 3, 7, 10]. In [6], the author defined a monopoly set of a graph G, proved that the $mo(G)$ for general graph is at least $\frac{n}{2}$, discussed the relationship between matchings and monopolies and he showed that any graph G admits a monopoly with at most $\alpha^{'}(G)$ vertices.

A monopoly set D of a graph G is called a connected monopoly set of G if the subgraph $\langle D \rangle$ induced by D is connected. The minimum cardinalities of connected monopolies sets of G, denoted by $cmo(G)$, is called the connected monopoly size of G. In this paper, we introduce and study the connected monopoly size of graphs and we investigate the relationship between $cmo(G)$ and some other parameters of a graph. Bounds for $cmo(G)$ and its exact values for some standard graphs are found. It is clear that, a connected monopoly size of a graph G is exists if and only if G is connected. Then we consider all graphs in this paper is connected, unless refer to otherwise. To illustrate this concept, consider the following graph G in Figure 1.

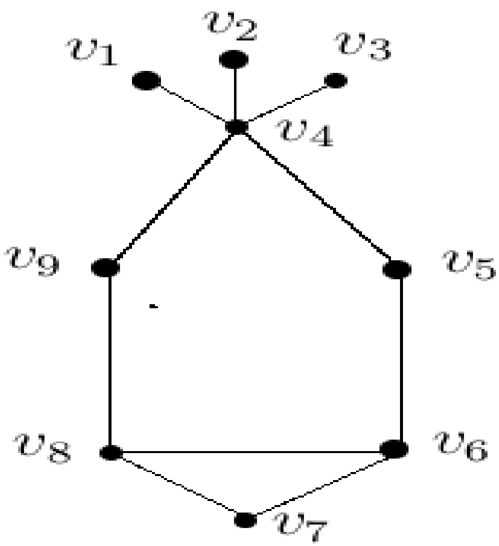


Fig 1: A graph G with $\gamma = 2$, $\gamma_c = 3$, $mo = 3$, $cmo = 4$.

The set $\{v_4, v_6\}$ is a dominating set of a graph G with minimum cardinality, then $\gamma(G) = 2$, the set $\{v_4, v_5, v_6\}$ is a connected dominating set of G with minimum cardinality, then $\gamma_c(G) = 3$, the set $\{v_4, v_6, v_7\}$ is a monopoly set of G with minimum cardinality, then $mo(G) = 3$ and the set $\{v_4, v_5, v_6, v_7\}$ is a connected monopoly set of G with minimum cardinality, then $cmo(G) = 4$.

2. Exact Values of Connected Monopoly Size of Some Standard Graphs

The connected monopoly size of some standard graphs can be easily found and are given as follows:

Observation 2.1

1. $cmo(K_n) = \lfloor \frac{n}{2} \rfloor$, $n \geq 2$.
2. $cmo(P_n) = cmo(C_n) = n - 2$, $n \geq 3$.
3. $cmo(K_{1,n}) = 1$, $n \geq 2$.
4. $cmo(K_{s,r}) = s + 1, \quad 2 \leq s \leq r$.
5. $$cmo(W_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & if \quad n \equiv 1 \ (\text{mod } 3); \\ \lceil \frac{n}{3} \rceil + 1, & otherwise \end{cases}.$$

Where W_n is a wheel graph of order $n \geq 4$.

Theorem 2.2 *Let T be a tree with n vertices. Then*

$$cmo(T) = n - l(T).$$

Where $l(T)$ is the number of pendent vertices in T.

Proof. Let T be a tree with n vertices, $L(T) = \{v \in T : d(v) = 1\}$ and let $l(T) = |L(T)|$. Then since for every $v \in L(T)$, $|N(v) \cap (T - L(T))| = 1 \geq \frac{d(v)}{2}$, it follows that $T - L(T)$ is a monopoly set of T. Furthermore, $\langle T - L(T) \rangle$ is a connected induce subgraph of T. Therefore,

$$cmo(T) \leq |T - L(T)| = n - l(T) \qquad (1).$$

Conversely, since for every $v \in T - L(T)$ is a cut-vertex of a tree T, it follows that

$$cmo(T) \geq n - l(T) \qquad (2).$$

From equations (1) and (2) we get $cmo(T) = n - l(T).$

3. Bounds on Connected Monopoly Size of Graphs

Relationships between a connected monopoly size $cmo(G)$ and some other parameters of a graph G as a monopoly size $mo(G)$, a domination number $\gamma(G)$, a connected domination number $\gamma_c(G)$, an independent number $\alpha(G)$ and a connectivity number $\kappa(G)$ may be get its as following:

Since a connected monopoly set of graphs is necessarily a monopoly set, it follows that the following result is obvious.

Proposition 3.1 For any connected graph G,

$$mo(G) \leq cmo(G).$$

It is immediate observation, from definitions, that a connected monopoly set of a graph G is a connected dominating set. Then the following results proof is not hard to get it.

Proposition 3.2 For any connected graph G,

$$\gamma_c(G) \leq cmo(G).$$

The following results is immediate consequences of Proposition 3.2.

Corollary 3.3 For any connected graph G,

$$\gamma(G) \leq cmo(G).$$

Theorem 3.4 Let G be a connected graph of order n and let p an integer number such that $0 \leq p \leq \alpha(G) - 2$. If $\kappa(G) \geq \alpha(G) - p$, then

$$cmo(G) \leq n - \alpha(G) + p + 1.$$

Proof. Let G be a connected graph of order n and let $0 \leq p \leq \alpha(G) - 2$. If $\kappa(G) \geq \alpha(G) - p$, then the subgraph $\langle V - I \rangle$ induced by subset $V - I$ is connected, where I is an independent set of G with cardinality equals to $\alpha(G) - p - 1$. Since I is an independent set, it follows that $|N(v) \cap (V - I)| \geq \frac{d(v)}{2}$ for every $v \in I$. Hence, $V - I$ is a connected monopoly set of G. Therefore,

$$cmo(G) \leq |V - I| \leq n - \alpha(G) + p + 1.$$

If $p = 0$ then $cmo(G) \leq n - \alpha + 1$, this situation, we can observe it for example, in $K_{r,r}$, where $\kappa(K_{r,r}) = \alpha(K_{r,r}) = r$ and $cmo(K_{r,r}) = r + 1 = n - \alpha + 0 + 1$.

Theorem 3.5 *Let G be a connected graph with minimum degree δ. Then*

$$cmo(G) \geq \left\lceil \frac{\kappa(G)}{2} \right\rceil.$$

Proof. Let G be a connected graph with minimum degree δ and let D be a connected monopoly set of G. Since D is a connected monopoly set, it follows that

$$|D| \geq |N(v) \cap D| \geq \frac{d(v)}{2} \geq \frac{\delta}{2}.$$

for every $v \in V - D$. But $\kappa(G) \leq \delta$ for every connected graph. Hence,

$$cmo(G) = |D| \geq \frac{\kappa(G)}{2}.$$

Theorem 3.6 *Let D be a minimum monopoly set of a connected graph G. If $|V-D|\leq \kappa(G)-1$, then*
$cmo(G)=mo(G).$

Proof. Let D be a minimum monopoly set of G and let $|V-D|\leq \kappa(G)-1$. Then the subgraph $\langle D\rangle$ induced by D is connected. Hence, $cmo(G)\leq |D|=mo(G)$. But from Proposition 3.1. we have $mo(G)\leq cmo(G)$ for any connected graph. And this completes the proof.

The following results is immediate consequences of Theorem 3.6.

Corollary 3.7 *For any connected graph G, if $mo(G)\geq n-\kappa(G)+1$, then*
$cmo(G)=mo(G).$

Lemma 3.8 *For a connected graph G of order $n\geq 3$,*
$cmo(G)\leq n-p.$

Where p is the number of pendent vertices of G (Vertices of degree equal to 1).

Theorem 3.9 *Let G be a connected graph of order $n\geq 3$. Then*
$cmo(G)\leq n-2.$

The bound is sharp, P_n and C_n attainting it.

Proof. Let G be a connected graph of order $n\geq 3$, and let $L\subset G$ define as $L=\{v\in V(G): d(v)=1\}$. Then by Lemma 3.8. we have $cmo(G)\leq n-|L|$.

If $|L|\geq 2$, then the theorem is hold. Otherwise, if $|L|<2$, then we consider the following cases:

Case 1: $|L|=1$, namely $L=\{v\}$, Choose a vertex $u\in V(G)$, $u\neq v$ such that u is not a cut-vertex in G. It is clear that $u\notin N(v)$. Hence, the subgraph $\langle V-\{v,u\}\rangle$ is connected. Furthermore, $V-\{v,u\}$ is a monopoly set in G. Therefore, $cmo(G)\leq |V-\{v,u\}|=n-2$.

Case 2: $|L|=0$, then $d(v)\geq 2$ for every $v\in V(G)$. Choose subset $S\subset V(G)$ and $S=\{u,w\}$ such that $uw\in E(G)$ and neither u nor w is a cut-vertex in G. Hence, the subgraph $\langle V-S\rangle$ is connected. Furthermore, $|N(u)\cap S|=1\leq \frac{d(u)}{2}$. Similarly for w. Hence, $V-S$ is a connected monopoly set in G. Therefore, $cmo(G)\leq n-2$.

We have from Theorem 3.9. that $1\leq cmo(G)\leq n-2$. In the following result we characterize all graphs which attainting the lower bound.

Theorem 3.10 *Let G be a connected graph of order n. Then if G has only one universal vertex (vertex with degree equal to n-1) and all its other vertices have degrees at most 2, then* $cmo(G)=1$.

Proof. Let G be a connected graph with vertex set $V=\{v,v_1,v_2,\dots,v_{n-1}\}$ such that $d(v)=n-1$ and $d(v_i)\leq 2$ for every $1\leq i\leq n-1$ and let $D=\{v\}$. Then $|N(v_i)\cap D|=1\geq \frac{d(v_i)}{2}$, for every $1\leq i\leq n-1$. Hence, the set D is a connected monopoly set in G, and $cmo(G)=1$.

Theorem 3.11 *Let G be a connected graph with maximum degree $\Delta\leq 2$. Then*
$cmo(G)=\gamma_c(G).$

Proof. By Proposition 3.2. we have $\gamma_c(G)\leq cmo(G)$. Conversely, Let D be a connected dominating set of G. Then from definition we have that $\langle D\rangle$ is connected, and $|N(v)\cap D|\geq 1\geq \frac{\Delta}{2}\geq \frac{d(v)}{2}$ for every $v\in V-D$. Hence, D is a connected monopoly set of G. Therefore, $cmo(G)\leq \gamma_c(G)$. This completes the proof.

The converse of Theorem 3.11. not true. For example, $cmo(K_{1,n}) = \gamma_c(K_{1,n}) = 1$, while $\Delta(K_{1,n}) = n$ for $n > 2$.

Theorem 3.12 *Let G be a connected graph. Then*

$$cmo(G) \leq 3mo(G) - 2.$$

Proof. Let D be a monopoly set in G and let $c(D)$ be the number of components in the subgraph $\langle D \rangle$ induced by D. It is clear that $mo(G) \geq c(D)$. Since every monopoly set in G is a dominating set, it follows that there exists two components of $\langle D \rangle$ (namely, $c_i(D)$ and $c_j(D)$, $i \neq j$) such that $d(c_i(D), c_j(D)) \leq 3$. By adding the vertices in the path between $c_i(D)$ and $c_j(D)$ to set D decreases the number of components in $\langle D \rangle$ by one. This procedure can be repeated it until remain only one component in $\langle D \rangle$. Thus resulting in a connected monopoly set in G. it is clear that there exists at most $2(c(D)-1)$ vertices added to D to form a connected monopoly set. Hence

$$\begin{aligned} cmo(G) &\leq |D| + 2(c(D)-1) \\ &\leq mo(G) + 2(mo(G)-1) \\ &\leq 3mo(G) - 2. \end{aligned}$$

The bound is sharp, P_n and C_n attainting this bound.

Theorem 3.13 *Let G* be a connected graph of order n and size m. Then

$$com(G) \leq 2m - n.$$

The bound is sharp, P_n achieves it.

Proof. Since for any connected graph $n - 1 \leq m$ it follows by Theorem 3.9. that

$$cmo(G) \leq n - 2 \leq 2(n-1) - n \leq 2m - n.$$

References

1. E. Berger, Dynamic monopolies of constant size, Journal of Combinatorial Theory, Series B, 83(2001), 191-200.
2. J. Bermond, J. Bond, D. Peleg and S. Perennes, The power of small coalitions in graphs, Discrete Applied Mathematics, 127(2003), 399 - 414.
3. P. Flocchini, R. Kralovic, A. Roncato, P. Ruzicka and N. Santoro, On time versus size for monotone dynamic monopolies in regular topologies. Journal of Discrete Algorithms, 1(2003), 129 - 150.
4. F. Harary, Graph Theory, Addison Wesley, Massachusetts, 1969.
5. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
6. K. Khoshkhak, M. Nemati, H. Soltani, and M. Zaker, A study of monopoly in graphs, Graph and Combinatorial Mathematics, 29(2013), 1417 - 1427.
7. A. Mishra and S. B. Rao, Minimum monopoly in regular and tree graphs, Discrete Mathematics, 306(14)(2006), 1586-1594.
8. E. Sampathkumar and H. B. Walikar, The connected domination number of a graph, journal of mathematics and physics science, 13(1979), 607-613.
9. D. Peleg, Local majorities; coalitions and monopolies in graphs; a review, Theoretical Computer Science, 282(2002), 231- 257.
10. M. Zaker, On dynamic monopolies of graphs with general thresholds, Discrete Mathematics, 312(2012), 1136-1143.