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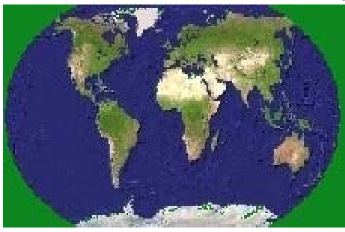

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M.A.Gopalan
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu,

S.Devibala
Asst.Professor, Department of Mathematics, Sri Meenakshi Govt.Arts College for Women, Madurai-625002, Tamil Nadu

D.Priyanka
Mphil Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

Correspondence:
M.A.Gopalan
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu,

On the ternary quadratic Diophantine equation $9z^2 = 11x^2 - 2y^2$

M.A.Gopalan, S.Devibala, D.Priyanka

Abstract
The ternary homogeneous quadratic equation given by $9z^2 = 11x^2 - 2y^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented. Also, given a solution, formulas for generating a sequence of solutions based on the given solutions are presented.

Keywords: Ternary quadratic, integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal numbers.

1. Introduction

Notations Used:

1. Polygonal number of rank 'n' with sides m
   a. $t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$
2. Stella octangular number of rank 'n'
   a. $SO_n = n(2n^2 - 1)$
3. Pyramidal number of rank 'n' sides m
   a. $P_n^m = \frac{n(n+1)}{6}[(m-2)n + (5-m)]$
4. Pronic number of rank 'n'
   $\Pr_n = n(n+1)$
5. Octahedral number of rank 'n'

$OH_n = \frac{1}{3}[n(2n^2+1)]$

Introduction:

The Diophantine equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-24] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $9z^2 = 11x^2 - 2y^2$ representing non- homogeneous cubic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

Method of Analysis:

The ternary quadratic Diophantine equation representing a cone under consideration is

$$9z^2 = 11x^2 - 2y^2 \quad (1)$$

To start with, it is seen that (1) is satisfied by the following non-zero integer triples (x, y, z): (27,45 , 21); (17, 35, 9); (29, 45, 21); ($2k^2$ +4k+11, $2k^2$ +22k+11, $2k^2$ -11); ($22k^2$ +4k+1, $22k^2$ +22k+1, $22k^2$ -1)

As the considered equation is symmetric in x,yand z , we have presented only positive integer solutions for clear understanding.

However, we have other choices of solution to (1) which are illustrated as below:

Choice 1:
Consider the linear transformations
$x = X + 2T$ (2)
$y = X + 11T$
Substituting (2)in(1),we get
$X^2 = z^2 + 22T^2$ (3)
which is satisfied by

$T = 2pq, X = q^2 + 22p^2$
$z = 22p^2 - q^2$ (4)
Substituting the values of X and T , we get the corresponding non-zero distinct integer solution to (1) given by
$x(p,q) = 22p^2 + q^2 + 4pq$
$y(p,q) = 24p^2 + 2q^2 + 24pq$ (5)
along with (4)

Properties:
A few interesting Properties obtained as follows:
- $23(y(q, q) - x(q, q)) = (23q)^2$
- $x(p, p) - z(p, p) = 6p^2$
- $x(p,p+1) + z(p,p+1) - 44t_{4,p} - 2t_{6,p} \equiv 0 \pmod 6$
- $x(p,q) + z(p,q) \equiv 0 \pmod 2$
- $x(p-1, p+1) + y(p-1, p+1) - z(p+1, p-1) - 56t_{4,p} \equiv 0 \pmod{40}$

Choice 2:
Rewrite (3) as
$$X^2 - z^2 = 22T^2$$
(*Factorizing (*),weget
(X+z) (X-z) =(22T)T
which is expressed in the form of ratio as
$$\frac{X+z}{22T} = \frac{T}{X-z} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (6)$$
This is equivalent to the system of double equations,
$$X\beta + z\beta - 22\alpha T = 0$$
$$\alpha z - X\alpha + \beta T = 0$$
Applying the method of cross multiplication and using (2), then the corresponding non-zero integer solution to (1) are obtained as
$x(\alpha,\beta) = 22\alpha^2 + \beta^2 + 4\alpha\beta$
$y(\alpha,\beta) = 22\alpha^2 + \beta^2 + 22\alpha\beta$
$z(\alpha,\beta) = 22\alpha^2 - \beta^2$

Properties:
A few interesting Properties obtained as follows:
- $6(x(\beta,\beta) - z(\beta,\beta)) = 6\beta^2$
- $2(x(\alpha,\alpha) - y(\alpha,\alpha)) = -36\alpha^2$
- $12(y(\alpha,\alpha^2) + x(\alpha,\alpha^2)) = (6\alpha)^3$
- $y(\alpha,\beta) + x(\alpha,\beta) \equiv 0 \pmod{18}$
- $z(\alpha,\alpha) = 21t_{4,\alpha}$

Remark:
In addition to (6), (3) may also be expressed in the form of ratios in following different ways that are presented below:

Way 1:
$$\frac{X+z}{T} = \frac{22T}{X-z} = \frac{\alpha}{\beta}$$
Way 2:
$$\frac{X+z}{2T} = \frac{11T}{X-z} = \frac{\alpha}{\beta}$$
Way 3:
$$\frac{X+z}{11T} = \frac{2T}{X-z} = \frac{\alpha}{\beta}$$
Solving each of the above system of equation by following the procedure as presented in choice 2, the corresponding integer solution to (1) are found to be as given below

Solution for way 1:
$x(\alpha,\beta) = 22\beta^2 + \alpha^2 + 4\alpha\beta$
$y(\alpha,\beta) = \alpha^2 + 22\beta^2 + 22\alpha\beta$
$z(\alpha,\beta) = 22\alpha^2 - \beta^2$

Solution for way 2:
$x(\alpha,\beta) = -2\alpha^2 - 11\beta^2 - 4\alpha\beta$
$y(\alpha,\beta) = -2\alpha^2 - 11\beta^2 - 22\alpha\beta$
$z(\alpha,\beta) = -2\alpha^2 + 11\beta^2$

Solution for way 3:
$x(\alpha,\beta) = -11\alpha^2 - 2\beta^2 - 4\alpha\beta$
$y(\alpha,\beta) = -11\alpha^2 - 2\beta^2 - 22\alpha\beta$
$z(\alpha,\beta) = -11\alpha^2 + 2\beta^2$

Choice 3:
(3) can be written as
$$z^2 + 22T^2 = X^2 = X^2 * 1$$
Assume
$X(a,b) = a^2 + 22b^2$ (8)
Write 1 as
$$1 = \frac{(3+i4\sqrt{22})(3-i4\sqrt{22})}{19^2} \quad (9)$$
Using (8) & (9) in (7) and applying the method of factorization define

$$(z+i\sqrt{22}T)=(a+i\sqrt{22}b)^2*\frac{(3+i4\sqrt{22})}{19}$$

Equating the real and imaginary parts and replacing a by 19A and b by 19B,we have

$$z=57A^2-3344AB-1254B^2 \quad (10)$$

$$T=76A^2+114AB-1672B^2 \quad (11)$$

From (8), (10), (11) & (2) we have

$$x(A,B)=513A^2+228AB-4598B^2$$

$$y(A,B)=1197A^2+1254AB-10450B^2$$

$$z(A,B)=57A^2-3344AB-1254B^2$$

which represent the non- zero distinct integer solutions to (1)

Properties:

A few interesting Properties obtained as follows:

- $21z(n(2n^2-1),1)-y(n(2n^2-1),1)=13083t_{3,1}+68970so_n$
- $21z((n+2),n(n+1))-y((n+2),n(n+1))=13083p^2{}_{r_n}+413820p_n{}^3$
- $9z(n^2,n)-x(n^2,n)=15884t^2_{2,n}+29868cp_{6,n}$
- $9z(n(n+1,1))-x(n(n+1,1))=29868ct_{2,n}-1748cp_{6,2}$
- $3y(1,1)-7x-836t_{2,1}=6(19)^2$

Note that, instead of (9), one may also write 1 as

$$1=\frac{(-3+i4\sqrt{22})(-3-i4\sqrt{22})}{19^2}$$

For this choice, the corresponding integer solutions, to (1) are given by

$$x(A,B)=316A^2-228AB+4598B^2$$

$$y(A,B)=1152A^2-1254AB-10450B^2$$

$$z(A,B)=-57A^2-3344AB+1254B^2$$

Choice 4:

Instead of (9), one may consider 1 as

$$1=\frac{(9+i2\sqrt{22})(9-i2\sqrt{22})}{13^2} \quad (12)$$

Using (12), (8) in (7) and applying the method factorization ,define

$$(z+i\sqrt{22}T)=(a+i\sqrt{22}b)^2*\frac{(9+i2\sqrt{22})}{13}$$

Equating the real and imaginary parts and replacing a by 19A and b by 19B, we have

$$z=117A^2-1114AB-2574B^2 \quad (13)$$

$$T=26A^2+234AB-572B^2 \quad (14)$$

From (8), (13),(14)&(2) we have

$$x(A,B)=413A^2+468AB+7828B^2$$

$$y(A,B)=647A^2+2574AB+1650B^2$$

$$z(A,B)=117A^2-114AB-2574B^2$$

Properties

- $x(A,A)-5806t_{5,A}\equiv 0(\mathrm{mod}\,2903)$
- $x(n(n+1),n)=1630t_{4,n}-466p^5{}_n$
- $y(1,pr_n)-z(1,pr_n)=106cs_2+3624pr^2{}_n+26688pr_n$

  $x(n(2n^2-1),1)-y(n(2n^2-1),1)+2z(n(2n^2-1),1)=1630t_{3,1}-2334s_{0,n}$
- $x(n(2n^2+1),1)-y(n(2n^2+1),1)+2z(n(2n^2+1),1)=326cs_2-23340H_n$

Note that, instead of (12), one may also write 1 as

$$1=\frac{(-9+i2\sqrt{22})(-9-i2\sqrt{22})}{13^2}$$

For this choice the corresponding integer solution to (1) are given by

$$x(A,B)=368A^2+6798B^2-468AB$$

$$y(A,B)=6020A^2-2574AB+1650B^2$$

$$z(A,B)=-117A^2-1144AB-2574B^2$$

Choice 5:

Consider (*) as the system of double equation

$$X+z=T^2 \quad (15)$$

$$X-z=22$$

Solving the above two equations ,we obtain

$$X=\frac{T^2+22}{2}$$

$$z=\frac{T^2-22}{2}$$

Since our interest is on finding integer solutions ,it is noted that the values of X & z are integers when T is even

In otherwords, choosing T=2K the corresponding non zero integer solutions to (1) are

$$x=2k^2+4k+11$$

$$y=2k^2+22k+11$$

$z = 2k^2 - 11$

Note:
Also (*) can be written in the system of equations as

1) $X + z = 11T$
$X - z = 2T$

2) $X + z = T^2$
$X - z = 22$

3) $X + z = 11T^2$
$X - z = 2$

Applying the procedure presented above one may obtain the integer solutions to (1)

Remarkable observations:

Let (x_0, y_0, z_0) be any given integer solution to (1)

Case (i):

Let
$$x_1 = h - 9x_0$$
$$y_1 = 9y_0 + h \qquad h \neq 0 \tag{16}$$
$$z_1 = 9z_0$$

be the second solutions of (1). Substituting (17) in (1) & performing a few calculations, we've

$h = 22x_0 + 4y_0$ and then

$$x_1 = 13x_0 + 4y_0$$
$$y_1 = 22x_0 + 13y_0$$

This is written in the form of matrix as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where

$$M = \begin{pmatrix} 13 & 4 \\ 22 & 13 \end{pmatrix} \tag{17}$$

Repeating the above process, the general solution (x_n, y_n) to (1) is given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{18}$$

To find M^n, the Eigen values of M are $\alpha = 1, \beta = 1$.
It is well-known that

$$M^n = \frac{\alpha^n}{\alpha - \beta}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I)$$

Using the above formula, we have

$$M^n = \frac{1}{2\sqrt{88}} \begin{pmatrix} \sqrt{88}(13+\sqrt{88})^n + \sqrt{88}(13-\sqrt{88})^n & (13+\sqrt{88})^n 4 - 4(13-\sqrt{88}) \\ 22(13+\sqrt{88})^n - 22(13+\sqrt{88})^n & \sqrt{88}(13+\sqrt{88})^n + \sqrt{88}(13-\sqrt{88})^n \end{pmatrix}$$

Thus the general solution (x_n, y_n, z_n) to (1) is given

$$x_n = \left[\frac{1}{2\sqrt{88}}[\sqrt{88}(13+\sqrt{88})^n + \sqrt{88}(13-\sqrt{88})^n]x_0 + [(13+\sqrt{88})^n 4 - 4(13-\sqrt{88})^n]y_0\right]$$

$$y_n = \frac{1}{2\sqrt{88}}\left[[22(13+\sqrt{88})^n - 22(13+\sqrt{88})^n]x_0 + [\sqrt{88}(13+\sqrt{88})^n + \sqrt{88}(13-\sqrt{88})^n]y_0\right] \quad z_n = 9^n z_0$$

Case (ii):

Let
$$x_1 = h - x_0$$
$$y_1 = y_0 \quad , \qquad h \neq 0$$
$$z_1 = z_0 + h$$

Repeating the process as in the case (i), the corresponding general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = \frac{1}{2\sqrt{99}}\left[[\sqrt{99}(10+\sqrt{99})^n + \sqrt{99}(10-\sqrt{99})^n]x_0 + [9(10+\sqrt{99})^n - 9(10-\sqrt{99})^n]y_0\right] \quad y_n = y_0$$

$$z_n = \frac{1}{2\sqrt{99}}\left[[11(10+\sqrt{99})^n - 11(10-\sqrt{99})^n]x_0 + [\sqrt{99}(10+\sqrt{99})^n + \sqrt{99}(10-\sqrt{99})^n]y_0\right]$$

Case (iii):

Let
$$x_1 = 11x_0$$
$$y_1 = 11y_0 - h \qquad h \neq 0$$
$$z_1 = 11z_0 + h$$

Repeating the process as in the case (i), the corresponding general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = (11)^n x_0$$

$$y_n = \frac{1}{22}\{[18(11)^n + 4(-11)^n]y_0 + [18(11)^n - 18(-11)^n)z_0]\}$$

$$z_n = \frac{1}{22}\{[4(11)^n - 4(-11)^n]y_0 + [4(11)^n + 18(-11)^n]z_0]\}$$

Conclusion:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by

$9z^2 = 11x^2 - 2y^2$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their Properties through special numbers.

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