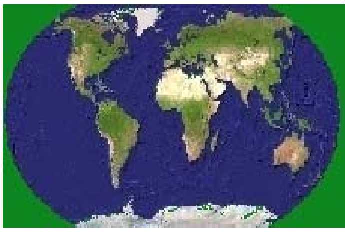
International Journal of Multidisciplinary Research and Development



International Journal of Multidisciplinary Research and Development

Volume :2, Issue :4, 211-214
April 2015
www.allsubjectjournal.com
e-ISSN: 2349-4182
p-ISSN: 2349-5979
Impact Factor: 3.762

M.A.Gopalan
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

T.Geetha
Lecturer, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

K.Hemalatha
Mphil Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

Correspondence:
M.A.Gopalan
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

On the ternary quadratic diophantine equation $5(x^2+y^2)-2xy=20z^2$

M.A.Gopalan, T.Geetha, K.Hemalatha

Abstract
The ternary homogeneous quadratic equation give by $5(x^2+y^2)-2xy=20z^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented. Also, given a solution, formulas for generating a sequence of solutions based on the given solutions is presented.

Keywords: Ternary quadratic, integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal numbers.

Notations Used:

1. Polygonal number of rank 'n' with sides m

$$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$$

2. Stella octangular number of rank 'n'

$$SO_n = n(2n^2 - 1)$$

3. Pyramidal number of rank 'n' sides m

$$P_n^m = \frac{n(n+1)}{6}[(m-2)n + (5-m)]$$

4. Pronic number of rank 'n'

$$\Pr_n = n(n+1)$$

1. Introduction:

The Diophantine equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-23] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $5(x^2 + y^2) - 2xy = 20z^2$ representing non- homogeneous cubic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. Method of Analysis:

The ternary quadratic diophantine equation representing a cone under consideration is

$$5(x^2 + y^2) - 2xy = 20z^2 \tag{1}$$

To start with, it is seen that (1) is satisfied by the following non-zero integer triples

(x, y, z): (12, 10, 7); (4, 10, 5); ($6k^2$ +2k-2, 10k, $6k^2$ +1)

As the considered equation is symmetric in x, y and z , we have presented only positive integer solutions for clear understanding.

However, we have other choices of solutions to (1) which are illustrated as below:

The substitution of the linear transformations

x = u+v ; y = u-v (2)

In (1) give

x = u+v; y = u-v (2)
in (1) gives

$$2u^2 + 3v^2 = 5z^2 \quad (3)$$

Again introducing the transformations
u=S+3T ; v=S-2T (4)
in (1) gives

$$S^2 + 6T^2 = z^2 \quad (5)$$

Now, (5) is solved through different methods and thus, different patterns of solutions to (1) are obtained.

Method: 1
Write (5) as

$$6T^2 = z^2 - S^2 = (z - S)(z + S) \quad (6)$$

which is written in the form of ratio as

$$\frac{6T}{(z+S)} = \frac{(z-S)}{T} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (7)$$

This is equivalent to the following two equations

$$\left.\begin{array}{l} \alpha S - 6\beta T + \alpha z = 0 \\ \beta S + \alpha T - \beta z = 0 \end{array}\right\} \quad (8)$$

Applying the method of cross multiplication, the above system of equations is satisfied by

$$u = -\alpha^2 + 6\beta^2 + 6\alpha\beta$$
$$v = -\alpha^2 + 6\beta^2 - 4\alpha\beta$$
$$Z = \alpha^2 + 6\beta^2 \quad (9)$$

substituting the values of u and v in (2), we get

$$x = -2\alpha^2 + 12\beta^2 + 2\alpha\beta$$
$$y = 10\alpha\beta \quad (10)$$

Thus (9) and (10) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-
A few interesting properties are as follows:-
(1) $x(a,a) - y(a,a) - 2t_{4,a} = 0$
(2) $3z(a,1) - z(a,1) - 2t_{4,a} - 12 = 0$
(3) $y(a, 2a^2 - 1) - 10\,So_a = 0$
(4) $y(\beta, 40\beta)$ Is a perfect square.
(5) $2z(a,1) - x(a,1) - y(a,1) - 16t_{4,a} + 12\,pra = 0$

Choice (ii)
(6) Is written in the form of ratio as

$$\frac{6T}{(z-S)} = \frac{(z+S)}{T} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (11)$$

which is equivalent to the following two equations

$$\left.\begin{array}{l} \alpha S + 6\beta T - \alpha z = 0 \\ \beta S - \alpha T + \beta z = 0 \end{array}\right\} \quad (12)$$

Applying the method of cross multiplication, the above system of equations is satisfied by

$$u = -\alpha^2 + 6\beta^2 - 6\alpha\beta$$
$$v = -\alpha^2 + 6\beta^2 + 4\alpha\beta$$
$$z = -\alpha^2 - 6\beta^2 \quad (13)$$

substituting the values of u and v in (2), we get

$$x = -2\alpha^2 + 12\beta^2 - 2\alpha\beta$$
$$y = -10\alpha\beta \quad (14)$$

Thus (13) and (14) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-
A few interesting properties are as follows:-
(1) $x(a,a) + y(a,a) + 2t_{4,a} = 0$
(2) $2z(a,1) + z(a,1) + 3t_{4,a} + 18 = 0$
(3) $y(10\beta, -\beta)$ is a perfect square.
(4) $y(\beta, -40\beta)$ is a perfect square.
(5) $x(1,a) + y(1,a) - 2z(1,a) - 36t_{4,a} + 12\,pra = 0$

Choice (iii)
Also (6) is written in the form of ratio as

$$\frac{2T}{(z-S)} = \frac{(z+S)}{3T} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (15)$$

which is equivalent to the following two equations

$$\left.\begin{array}{l} \alpha S + 2\beta T - \alpha z = 0 \\ \beta S - 3\alpha T + \beta z = 0 \end{array}\right\} \quad (16)$$

Applying the method of cross multiplication, the above system of equations is satisfied by

$$u = -3\alpha^2 + 2\beta^2 - 6\alpha\beta$$
$$v = -3\alpha^2 + 2\beta^2 + 4\alpha\beta$$
$$z = -3\alpha^2 - 2\beta^2 \quad (17)$$

Satisfied the values of u and v in (2), we get

$$x = -6\alpha^2 + 4\beta^2 - 2\alpha\beta$$
$$y = -10\alpha\beta \quad (18)$$

Thus (17) and (18) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-
A few interesting properties are as follows:-
(1) $x(a,a) - y(a,a) - 6t_{4,a} = 0$
(2) $6z(1,a) - z(1,a) + 10t_{4,a} + 15 = 0$
(3) $x(2a,a) - y(2a,a) + 36t_{4,a} = 0$
(4) $y(a, 4a^2 - 3) - 10\,So_a = 0$
(5) $y(\beta, 90\beta)$ is a perfect square

Choice (iv)
Again (6) is written in the form of ratio as

$$\frac{2T}{(z+S)} = \frac{(z-S)}{3T} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (19)$$

This is equivalent to the following two equations

$$\alpha S - 2\beta T - \alpha z = 0 \; ; \; \beta S + 3\alpha T - \beta z = 0 \quad (20)$$

Applying the method of cross multiplication, the above system of equations is satisfied by

$$u = -3\alpha^2 + 2\beta^2 + 6\alpha\beta$$
$$v = -3\alpha^2 + 2\beta^2 - 4\alpha\beta$$

z = $3\alpha^2 + 2\beta^2$ (21)

substituting the values of u and v in (2), we get

$$x = -6\alpha^2 + 4\beta^2 + 2\alpha\beta$$
$$y = 10\alpha\beta \quad (22)$$

Thus (21) and (22) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(a,a) - y(a,a) + 10t_{4,a} = 0$

(2) $2z(a,1) - z(a,1) - 3t_{4,a} - 2 = 0$

(3) $x(3a,a) - y(3a,a) + 74t_{4,a} = 0$

(4) $y(9a^2 - 8, a) - 10\,So_a = 0$

(5) $y(160\beta, \beta)$ is a perfect square.

Method: 2

Equation (4) is written as

$$S^2 + 6T^2 = z^2 = z^2 * 1 \quad (23)$$

Assume $z = z(a,b) = a^2 + 6b^2$ (24)

where a,b are non-zero integers,

write 1 as

$$1 = \frac{(1 + i2\sqrt{6})}{5}\frac{(1 - i2\sqrt{6})}{5} \quad (25)$$

substituting (24) and (25) in (23) it is written in the factorizable form as

$$(S+i\sqrt{6}T)(S-i\sqrt{6}T) = (a+i\sqrt{6}b)^2(a-i\sqrt{6}b)^2 * \frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{5^2}$$

Equating the positive and negative factors, we get

$$(S + i\sqrt{6}T) = (a + i\sqrt{6}b)^2 * \frac{(1 + i2\sqrt{6})}{5} \quad (26)$$

$$(S - i\sqrt{6}T) = (a - i\sqrt{6}b)^2 * \frac{(1 - i2\sqrt{6})}{5} \quad (27)$$

Equating real and imaginary parts either in (26) or (27) and replacing a by 5A, b by 5B we get

u= $35A^2 - 210B^2 - 90AB$

v= $-15A^2 + 90B^2 - 140AB$

z = $25(A^2 + 6B^2)$ (28)

Also from (24) the values of the u and v in (2), we've

x= $x(a, b) = 20A^2 - 120B^2 - 230AB$

y= $y(a, b) = 50A^2 - 300B^2 + 50AB$ (29)

Thus (28) and (29) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(a,a) - y(a,a) + 130t_{4,a} = 0$

(2) $4z(1,a) - z(1,a) - 18t_{4,a} - 3 = 0$

(3) $z(2\beta, 4\beta)$ is a perfect square.

(4) $z(5\beta, 10\beta)$ is a perfect square.

(5) $x(a,a) + y(a,a) - z(a,a) + 537t_{4,a} = 0$

3. Conclusion:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by

$$5(x^2 + y^2) - 2xy = 20z^2$$

As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

Acknowledgement:

The financial support from the UGC, New Delhi (F-MRP-5122/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged.

5. **Reference:**

1. Dickson LE.History of Theory of Numbers and Diophantine Analysis,Vol 2,Dove publications,New York 2005.
2. Mordell LJ. Diophantine Equations" Academic Press, Newyork 1970.
3. Carmicheal RD.The Theory of Numbers and Diophantine Analysis , Dover publications,,Newyork 1959.
4. Gopalan MA,Geetha D. Lattice points on the Hyperboloid of two sheets $x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4$ Impact J Sci Tech 2010;4:23-32.
5. Gopalan MA ,Vidhyalakshmi S,Kavitha A,Integral points on the Homogeneous cone $z^2 = 2x^2 - 7y^2$,The Diophantus J Math 2012;1(2):127-136.
6. Gopalan MA,Vidhyalakshmi S, Sumathi G.Lattice points on the Hyperboloid of one sheet $4z^2 = 2x^2 + 3y^2 - 4$. Diophantus J Math 2012; 1(2): 109-115.
7. Gopalan MA,Vidhyalakshmi S,Lakshmi K. Integral points on the Hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$. Diophantus J Math 2012; 1(2): 99-107.
8. Gopalan MA,Vidhyalakshmi S, Mallika S.Observations on Hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 2$. Bessel J Math 2012; 2(3): 221-226.
9. Gopalan MA ,Vidhyalakshmi S,Usha Rani TR , Mallika S,Integral points on the Homogeneous cone $6z^2 + 3y^2 - 2x^2 = 0$,The Impact J Sci Tech 2012;6(1):7-13.
10. Gopalan MA ,Vidhyalakshmi S,Sumathi G,Lattice points on the Elliptic parabolid $z = 9x^2 + 4y^2$,Advances in Theoretical and Applied Mathematics 2012;m7(4):379-385
11. Gopalan MA ,Vidhyalakshmi S,Usha Rani TR,Integral points on the non- homogeneous cone $2z^2 + 4xy + 8x - 4z = 0$, Global Journal of Mathamatics and Mathamatical sciences 2012;2(1):61-67

12. Gopalan MA ,Vidhyalakshmi S,Lakshmi K.,Lattice points on the Elliptic paraboloid $16y^2 + 9z^2 = 4x$, Bessel J of Math 2013; 3(2): 137-145.
13. Gopalan MA ,Vidhyalakshmi S,Uma Rani J , Integral points on the Homogeneous cone $4y^2 + x^2 = 37z^2$,Cayley J of Math 2013;2(2):101- 107.
14. Gopalan MA,Vidhyalakshmi S, Kavitha A. Observations on the Hyperboloid of two sheets $7x^2 - 3y^2 = z^2 + z(y - x) + 4$. International Journal of Latest Research in Science and technology 2013; 2(2): 84-86.
15. Gopalan MA, Sivagami B. Integral points on the homogeneous cone $z^2 = 3x^2 + 6y^2$. ISOR Journal of Mathematics 2013; 8(4): 24-29.
16. Gopalan MA,Geetha V. Lattice points on the homogeneous cone $z^2 = 2x^2 + 8y^2 - 6xy$. Indian journal of Science 2013; 2: 93-96.
17. Gopalan MA, Vidhyalakshmi S ,Maheswari D. Integral points on the homogeneous cone $35z^2 = 2x^2 + 3y^2$. Indian journal of Science 2014; 7: 6-10.
18. Gopalan MA, Vidhyalakshmi S, Umarani J. On the Ternary Quadratic Diophantine Equation $6(x^2 + y^2) - 8xy = 21z^2$. Sch J Eng Tech 2014; 2(2A): 108-112.
19. Meena K,Vidhyalakshmi S, Gopalan MA , Priya IK . Integral points on the cone $3(x^2 + y^2) - 5xy = 47z^2$. Bulletin of Mathematics and statistic Research 2014; 2(1): 65-70.
20. Gopalan MA, Vidhyalakshmi S ,Nivetha S.On Ternary Quadratic Diophantine Equation $4(x^2 + y^2) - 7xy = 31z^2$. Diophantus J Math 2014; 3(1): 1-7.
21. Meena K,Vidhyalakshmi S, Gopalan MA ,Thangam SA. Integral solutions on the homogeneous cone $28z^2 = 4x^2 + 3y^2$. Bulletin of Mathematics and statistic Research 2014; 2(1): 47-53.
22. Santhi J ,Gopalan MA, Vidhyalakshmi. Lattice points on the homogeneous cone $8(x^2 + y^2) - 15xy = 56z^2$. Sch Journal of Phy Math Stat 2014; 1(1): 29-32.
23. Meena S,Gopalan MA, Vidhyalakshmi S ,Thiruniraiselvi N.Observations on the Ternary Quadratic Diophantine Equation $x^2 + 9y^2 = 50z^2$.International Journal of Applied Research 2015; 1(2): 51-53.