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Solution of 1-dimensional Wave equation by Elzaki Transform

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Abstract
In this paper a new Integral transform, namely Elzaki Transform was applied to solve 1-dimensional wave equation to obtained the exact solutions. It is more simple, efficient, easier and powerful tool for solving 1-dimensional wave equation. The ability of this method is illustrated by means of example.

Keywords: partial differential equation, wave equation, elzaki transform

1. Introduction

The wave equation usually describes water waves, the vibrations of a string or a membrane, the propagation of electromagnetic and sound waves, or the transmission of electric signals in a cable. The function $u(x,t)$ defines a small displacement of any point of a vibrating string at position *x* at time *t*. The wave equation contains the term u_{tt} that represents the vertical acceleration of a vibrating string at point *x*, which is due to the tension in the string. The wave equation plays a significant role in various physical problems. The study of wave equation is needed in diverse areas of science and engineering.

Engineering, astronomy, physics and many more fields are capable of using integral transforms. The Laplace transform is varsity in use and application among all the transforms. Ordinary and partial differential equations are solved by integral transforms easily. In recent time Tariq Elzaki put forth a new transform known as Elzaki transform [6, 7, 8, 9] which is based on Fourier transform and further used it to the solution of ordinary differential equations [7], system of partial differential equations [8], telegraph equation and also integral equation [6]. This method advantages the others in the way that it eliminates the need of linearization, perturbation or any other transformation. It also reduces the mountainous computation work previously required by other methods [1, 2, 3, 4, 5].

Elzaki Transform

Definition [8]

Elzaki Transform of a function $f(t)$ is denoted by $E[f(t)]$ and is defined as follows

$$E[f(t)] = T(v) = v\int_0^\infty e^{-\frac{t}{v}} f(t)dt \qquad , \qquad v \in (-k_1, k_2)$$

We use integration by parts to obtain Elzaki Transform of partial derivatives as

$$E\left[\frac{\partial f(x,t)}{\partial t}\right] = v\int_0^\infty \frac{\partial f}{\partial t} e^{-\frac{t}{v}} dt = \lim_{m\to\infty}\int_0^m v\frac{\partial f}{\partial t} e^{-\frac{t}{v}} dt$$

$$= \lim_{m\to\infty}\left\{\left[ve^{-\frac{t}{v}} f(x,t)\right]_0^m - \int_0^m e^{-\frac{t}{v}} f(x,t)dt\right\} = \frac{T(x,v)}{v} - vf(x,0)$$

Also

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{T(x,v)}{v^2} - f(x,0) - vf_t(x,0)$$

We assume that *f* is piecewise and of exponential order

$$E\left[\frac{\partial f(x,t)}{\partial x}\right] = v\int_0^\infty \frac{\partial f}{\partial x} e^{-\frac{t}{v}} dt = \frac{\partial}{\partial x}\int_0^m ve^{-\frac{t}{v}} f(x,t)dt$$

Using the Leibnitz rule we find that

$$E\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}[T(x,v)]$$

We can also find

$$E\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)]$$

The Operation Properties of Elzaki Transform:-
ELzaki Transform of some Functions

Table 1

Sr. No	$f(t)$	$E[f(t)] = T(v)$
1	1	v^2
2	t	v^3
3	t^n	$n!\ t^{n+2}$
4	e^{at}	$\frac{v^2}{1-av}$
5	te^{at}	$\frac{v^3}{(1-av)^2}$
6	$\frac{t^{n-1}e^{at}}{(n-1)!}$, $n = 1, 2, \ldots$	$\frac{v^{n+1}}{(1-av)^n}$
7	$sinat$	$\frac{av^3}{1+a^2v^2}$
8	$cosat$	$\frac{v^2}{1+a^2v^2}$
9	$sinhat$	$\frac{av^3}{1-a^2v^2}$
10	$coshat$	$\frac{av^2}{1-a^2v^2}$
11	$e^{at}sinbt$	$\frac{bv^3}{(1-av)^2+b^2v^2}$
12	$e^{at}cosbt$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$
13	$tsinat$	$\frac{2av^4}{1+a^2v^2}$
14	$tcosat$	$\frac{v^3}{1+a^2v^2}$

Numerical Applications:
Example 1 [1]:

$$\frac{\partial u^2(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < \pi\ ,\ \ 0 < t \tag{1}$$

With boundary conditions

$$u(0,t) = u(\pi,t) = 0\ ,\quad 0 < t \tag{2}$$

And initial conditions

$$u(x,0) = \sin x\,,\quad u_t(x,0) = 0 \qquad 0 \le x \le \pi$$

Solution:
The exact solution is

$$u(x,t) = \cos t \sin x$$

Taking Elzaki Transform on both sides on (1) & (2) we get

$$\frac{T(x,v)}{v^2} - u(x,0) - vu_t(x,0) = \frac{d^2T(x,v)}{dx^2} \tag{3}$$

$$T(0,v) = T(\pi,v) = 0 \tag{4}$$

Using initial conditions we get

$$v^2\frac{d^2T(x,v)}{dx^2} - T(x,v) = -v^2\sin x$$

$$v^2T'' - T = -v^2\sin x$$

After solving above differential equation we get

$$T(x,v) = C_1e^{\frac{x}{v}} + C_2e^{-\frac{x}{v}} + \frac{v^2}{1+v^2}\sin x$$

Using (4) we get

$$T(x,v) = \frac{v^2}{1+v^2}\sin x$$

Taking inverse Elzaki Transform on both sides

$$E^{-1}(T(x,v)) = E^{-1}\left(\frac{v^2}{1+v^2}\sin x\right)$$

$$u(x,t) = \sin x\, E^{-1}\left(\frac{v^2}{1+v^2}\right)$$

$$u(x,t) = \cos t \sin x$$

Example 2 [1, 2]:

$$\frac{\partial u^2(x,t)}{\partial t^2} = \frac{1}{16\pi^2}\frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < 0.5\ ,\ \ 0 < t \tag{5}$$

With boundary conditions

$$u(0,t) = u(0.5,t) = 0\ ,\quad 0 < t \tag{6}$$

And initial conditions

$$u(x,0) = 0\ ,\ \ u_t(x,0) = \sin 4\pi x \qquad ,0 \le x \le 0.5$$

Solution:
The exact solution is

$$u(x,t) = \sin t \sin 4\pi x$$

Taking Elzaki Transform on both sides on (5) & (6) we get

$$\frac{T(x,v)}{v^2} - u(x,0) - vu_t(x,0) = \frac{1}{16\pi^2}\frac{d^2T(x,v)}{dx^2} \quad (7)$$

$$T(0,v) = T(0.5,v) = 0 \quad (8)$$

Using initial conditions on (7) we get

$$\frac{v^2}{16\pi^2}\frac{d^2T(x,v)}{dx^2} - T(x,v) = -v^3 \sin 4\pi x$$

$$\frac{v^2}{16\pi^2}T'' - T = -v^3 \sin 4\pi x$$

After solving above differential equation we get

$$T(x,v) = C_1 e^{\frac{4\pi x}{v}} + C_2 e^{-\frac{4\pi x}{v}} + \frac{v^3}{1+v^2}\sin 4\pi x$$

Using (8) we get

$$T(x,v) = \frac{v^3}{1+v^2}\sin 4\pi x$$

Taking inverse Elzaki Transform on both sides

$$E^{-1}(T(x,v)) = E^{-1}\left(\frac{v^3}{1+v^2}\sin 4\pi x\right)$$

$$u(x,t) = \sin 4\pi x \, E^{-1}\left(\frac{v^3}{1+v^2}\right)$$

$$u(x,t) = \sin t \sin 4\pi x$$

Example 3 [2]:

$$\frac{\partial u^2(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < \pi\,, \quad 0 < t \quad (9)$$

With boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 1\,, \quad 0 < t \quad (10)$$

And initial conditions

$$u(x,0) = x\,, \qquad u_t(x,0) = cosx$$

Solution:
The exact solution is

$$u(x,t) = x + \cos x \sin \mathrm{t}$$

The given Neumann boundary conditions $u_x(0,t)$ and $u_x(\pi,t)$ are inhomogeneous.

Taking Elzaki Transform on both sides on (9) & (10) we get

$$\frac{T(x,v)}{v^2} - u(x,0) - vu_t(x,0) = \frac{d^2T(x,v)}{dx^2} \quad (11)$$

$$T'(0,v) = T'(\pi,v) = v^2 \quad (12)$$

Using initial conditions on (11) we get

$$v^2\frac{d^2T(x,v)}{dx^2} - T(x,v) = -v^2x - v^3\cos x$$

$$v^2T'' - T = -v^2x - v^2\cos x$$

After solving above differential equation we get

$$T(x,v) = C_1 e^{\frac{x}{v}} + C_2 e^{-\frac{x}{v}} + v^2x + \frac{v^3}{1+v^2}\cos x$$

Using (12) we get

$$T(x,v) = v^2x + \frac{v^3}{1+v^2}\cos x$$

Taking inverse Elzaki Transform on both sides

$$E^{-1}(T(x,v)) = E^{-1}\left(v^2x + \frac{v^3}{1+v^2}\cos x\right)$$

$$u(x,t) = xE^{-1}(v^2) + \cos x \, E^{-1}\left(\frac{v^3}{1+v^2}\right)$$

$$u(x,t) = x + \cos x \sin \mathrm{t}$$

Conclusion

In this work, 1-dimensional wave equation is solved by Elzaki Transform. This result has been extracted that Elzaki Transform plays a key role in finding the analytical solution for a wide class of initial boundary value problems. Its use in PDEs and ODEs is heavily. The utilization of this method is simple in use, economic, time saving and exquisite. This method giving better solution than the other existing methods.

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