International Journal of Multidisciplinary Research and Development 2014; 1(7): 469-472

International Journal of Multidisciplinary Research and Development

IJMRD 2014; 1(7): 469-472
www.allsubjectjournal.com
Received: 25-12-2014
Accepted: 27-01-2015
E-ISSN: 2349-4182
P-ISSN: 2349-5979
Impact factor: 3.762

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A brief introduction about the partition theory

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Abstract
In this paper, we introduce Partition theory and 1974 CONJECTURE 1 OF ANDREWS on two three-parameter partition functions $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$. Conjecture 1 for k = a+1 and show that it is false for k ≥ a+2.
Further, we explain an introduction of Schur's 1926 theorem on partitions. We also introduce 1974 CONJECTURE 2 OF ANDREWS on partitions.
Further, we state Bressound's conjecture, which states that two sets of partitions under certain constraints are equinumerous. The validity of the conjecture in the first two cases implies exactly the partition-theoretical interpretation for the Rogers-Ramanujan identities.

Keywords: partition theory, $A_{\lambda,k,a}(n)$, $B_{\lambda,k,a}(n)$

An Introduction Mathematical theory of Partitions
Partition theory is a fundamental area of number theory. It is concerned with the number of ways that a whole number can be partitioned into whole number parts.
Take a positive integer number, say 5 and write it as a sum of smaller or equal positive integers:

5 = 5	We therefore have
= 4+1	7 «ordered» partitions of
= 3+2	the number 5
= 3+1+1	Ordered means that we
= 2+2+1	always start with the
= 2+1+1+1	biggest number, e.g.
= 1+1+1+1+1	we do not count 4+1 and 1+4 as two different Partitions

In other words,
for example 5 can be partitioned in 7 ways thus: 11111, 2111, 221, 311, 32, 41, 5.
The permutations of these 7 partitions add up to 16 thus:

11111	= 1 permutation
2111	= 4 permutations
221	= 1 permutation
311	= 3 permutations
32	= 2 permutations
41	= 2 permutations
5	= 3permutation

A partition is a way of writing an integer n as a sum of positive integers where the order of the parts is not significant.

- Each of the sums is a partition of 5. The partition 4+1 is a partition of 5 into two distinct parts. Moreover, this partition has length 2, since it has two parts.
- Partitions can be represented by using diagrams which are called Ferrer's diagrams for example for the number 4:

4 = 3 +1 = 2 + 2 = 2 + 1 +1 = 1 + 1+ 1+ 1

1+1+1+1=2+1+1=2+2 = 3+1 =4

- However, there are other ways of studying partitions and one of them is to use generating functions.
- In general, a generating function for a sequence of numbers a_0, a_1, a_2, …. is defined as:

$$G(x) = \sum_{k=0}^{\infty} x^k a^k$$

- For example, the generating function of non-negative integer numbers {0,1,2…} is

$$G_1(x) = \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

- The last equality follows by using Taylor's expansion
- Euler discovered that one could define generating functions for the number of partitions of a positive integer.
- It is possible to use generating functions in order to prove some identities between partitions. for example one can show the so-called Euler's parity law: the number of partitions of a number n into distinct parts equals the number of partitions of the same number into odd parts.

If we take the set A to be the set of all positive integers, with the a_i's unrestricted, where repetitions are allowed, we have the standard definition for a partition of a positive integer.

A partition of a non-negative integer n is a representation of n as a sum of positive integers, called summands or parts. We will denote the total number of partitions of n by P(n). P(O) is defined to be 1. G. W. Liebnitz (1646 - 1716) was among the first mathematicians who paid attention in the development stages in this area of mathematics (Griffin, 1954), but the greatest contributions in the early stages of partition theory were due to L. Euler (1707-1783) (according to Andrews, 1971). Over the centuries a great number of mathematicians had devoted their time in a searchfor new identities in partition theory and to find a formula for P(n).Further developments in partition theory in the early 20th century were due to G. H. Hardy (I877 - 1947) and S. Ramanujan (I887 -1920), and in 1917 they showed that P(n) is asymptotic to $\frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$ (Cohen,1978).

After that we study of two three-parameter functions $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$ which functions are defined as bellow:

Definition

For an even integer λ, we denote $A_{\lambda,k,a}(n)$ the number of partitions of n into parts such that no part ≠ 0 (mod λ+1) may be repeated and no part is $\equiv 0, \pm(2a-\lambda)\frac{1}{2}(\lambda+1)$[mod (2k - λ+1) (λ + 1)]. If λ is an odd integer, we denote by $A_{\lambda,k,a}(n)$ the number of partitions of n into parts such that no part ≠ 0 [mod $\frac{1}{2}(\lambda+1)$] may be repeated, no part is ≡ λ+1 [mod (2λ+2)] and no part is ≡ $0, \pm(2a-\lambda)\frac{1}{2}(\lambda+1)$

The generatinfunctionfor $A_{\lambda,k,a}$ (n) is given by

$$1 + \sum_{n=1}^{\infty} A_{\lambda,n,a}(n)\, Q^n = \prod_{j=1}^{\infty}(1+Q^j)(1+Q^{j(\lambda+1)})^{-1}(1-Q^{j(\lambda+1)})^{-1}$$
$$X\,[1 - Q^{(2k-\lambda+1)(\lambda+1)j-\left(a-\frac{\lambda}{2}\right)(\lambda+1)}]$$
$$X[1 - Q^{(2k-\lambda+1)(\lambda+1)(j-1)+\left(a-\frac{\lambda}{2}\right)(\lambda+1)}]$$
$$X\,[1 - Q^{(2k-\lambda+1)(\lambda+1)j}]$$

Definition

For an integer λ, we denote $B_{\lambda,k,a}(n)$ the number of partitions of type $b_1+ - - -+b_s$ with $b_i \geq b_{i+1}$ no parts $\neq 0(\text{mod } \lambda +1)$ is repeated, $b_i - b_{i+k-1} \geq \lambda + 1$ with strict inequality if $(\lambda+l) / b_i$ and finally if f_j denotes the number of appearances of j in the partition, then

$$\sum_{i=j}^{\lambda-j+1} f_j \leq a - j$$

for $1 \leq j \leq \frac{1}{2}(\lambda+1)$ and $f_1 + \ldots\ldots + f_{\lambda+1} \leq a-1$ where f is the number of appearances of j in the partition. The Computer Andrews [1] suggest

Conjecture 1

For $\frac{\lambda}{2} \leq \alpha \leq k < \lambda$,

Let $n^C = \frac{(k+\lambda-a+1)(k+\lambda-a)}{2} + (k-\lambda+1)(\lambda+1)$. Then

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) \text{ for } 0 \leq n < n^c$$

and $$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + 1 \quad \text{for } n = n^c.$$

Which is true for $k = a + 1$ and it is false for $k \geq a + 2$.
We get a different combinatorial proof of a particular case of the following generalized version of Schur's theorem.

Schur's theorem

Given positive integers r and m such that $r < \frac{m}{2}$ let $C_{r,m}(n)$ denote the number of partitions of n into distinct parts $\equiv \pm r(\text{mod } m)$ and let $D_{r,m}(n)$ denote the number of partitions of n into distinct parts $\equiv 0, \pm r(\text{mod } m)$ with minimal difference m, minimal difference 2m between multiples of m. Then $C_{r,m}(n) = D_{r,m}(n)$ for all n.
In 1974, Andrews conjectured the following Conjecture:

Conjecture 2

for all $n \geq 0$, We have

$$A_{4,3,3}(n) = B^0_{4,3,3}(n)$$

where $B^0_{4,3,3}(n)$ is the number of partitions enumerated by with the $B_{4,3,3}(n)$ added conditions

$$f_{5j+2} + f_{5j+3} \leq 1 \text{ for } j \geq 0,$$

$$f_{5j+4} + f_{5j+6} \leq 1 \text{ for } j \geq 0,$$

$$f_{5j-1} + f_{5j} + f_{5j+5} + f_{5j+6} \leq 3 \text{ for } j \geq 1,$$

where, as before, f_j denotes the number of appearances of j in the partition.
G.E. Andrews, C. Bessenrodt and J.B.Olsson have proved conjecture 2 analytically. In Bressound's conjecture, which states that two sets of partitions under certain constraints are equinumerous.

Theorem (Conjectured by D. M. Bressoud).

For any $n \geq 0$, we have $A_{a,b}(n) = B_{a,b}(n)$.

In a quite different direction, using sieve methods, Andrews connected yet another set of partitions to the above mentioned families, and Bressoud[8] extended this work to all moduli. For any positive integer $a > b$ with $a + b \geq 5$, let $Q_{a,b}(n)$ denote the number of partitions of n with all the successive ranks bounded in $[- b + 2 , a - 2]$. Then we have the following Andrews-Bressoud Successive Rank theorem.

Theorem For any $n \geq 0$, we have

$$A_{a,\,b}(n) = Q_{a,,b}(n).$$

References

1. Andrews GE. On the General Rogers - Ramanujan Theorem, Mem. Amer. Math. Soc., No. 1974; 152:1-86.
2. George E. Andrews, Kimmo Eriksson Integer Partitions. Cambridge University Press, 2004. ISBN 0-521-60090-1.
3. Bressoud DM. A combinatorial proof of Schur's 1926 partition theorem, Proc. Amer. Math. Soc., 1980; 79(2):338-340.
4. Andrews GE, Bessenrodt C, Olsson JB. Partition Identities and Labels for Modular Characters, Trans. Amer. Math. Soc. 1994; 344 (2):597-615.
5. David M. Bressoud, Extension of the partition sieve, J. Number Theory 1980; 12(1):87-100. MR 566873 (82e:10022)
6. Padmavathamma, Sudha TG. A Note on some Partition Identities, Bull. Cal. Math. Soc., 1993; 85:403-408.
7. Schur IJ, Zur additiven Zahlentheorie, Sitzungsber. Deutsch. Adak. Wissensch. Berlin, Phys - Math. Kl, 1926, 488-495.
8. George E. Andrews, The Theory of Partitions Cambridge University Press, 1976, ISBN 0-521-63766-X.
9. Hardy GH, Wright EM. [1938]. An Introduction to the Theory of Numbers, 2008.
10. David M Bressoud. A generalization of the Rogers-Ramanujan identities for all moduli, J. Combin. Theory Ser. A 1979; 27(1):64-68. MR 541344 (81c:05008).