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On the integer solutions of the pell equation $x^2 = 20y^2 - 4^t$

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Abstract

The binary quadratic Diophantine equation represented by $x^2 = 20y^2 - 4^t$, $t > 0$ is considered and analyzed for its non-zero distinct integer solutions for the choices of t given by (i) $t = 1$ (ii) $t = 2k$ and (iii) $t = 2k + 1$. A few interesting relations among the solutions are presented. Further, recurrence relations on the solutions are obtained.

Keywords: Pell equation, Integer solutions, Binary quadratic Diophantine equation.

1. Introduction

Pell's equation is any Diophantine equation of the form $x^2 - ny^2 = 1$, when n a given positive non-square integer is has always positive integer solutions. This equation was first studied extensively in India, starting with Brahmagupta, who developed the Chakravala method to Pell's equation and other quadratic indeterminate equations. When k is a positive integer and $n \in (k^2 \pm 4, k^2 \pm 1)$, positive integer solutions of the equations $x^2 - ny^2 = \pm 4$ and $x^2 - ny^2 = \pm 1$, have been investigated by Jones in [2]. In [1, 4, 6, 7, 8, 9] and [11]. Some special Pell equation and their solutions are considered. In [3]. The integer solutions of the Pell equation $x^2 - dy^2 = 2^t$ has been considered. In [5], the Pell equation $x^2 - (k^2 - k)y^2 = 2^t$ is analyzed for its integer solutions. In [10], the Pell equation $x^2 - 3y^2 = (k^2 + 4k + 1)^t$ is analyzed for its positive integer solutions.

In this communication, we present the Pell equation $x^2 = 20y^2 - 4^t$, where $t > 0$ and infinitely many positive integer solutions are obtained for the choices of t given by (i) $t = 1$ (ii) $t = 2k$ and (iii) $t = 2k + 1$. A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are obtained.

2. Notations

$t_{4,n}$ = Square number of rank n .

3. Method of Analysis

$t = 1$

The Pell equation is

$$x^2 = 20y^2 - 4 \tag{1}$$

Let (x_0, y_0) be the initial solutions of equation (1), given by

$$x_0 = 4;\ y_0 = 1.$$

To find the other solutions of equation (1), Consider the Pellian equation

$$x^2 = 20y^2 + 1 \tag{2}$$

Here $\tilde{x}_0 = 9;\ \tilde{y}_0 = 2.$

Whose initial solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2} f_n$$

$$\tilde{y}_n = \frac{1}{4\sqrt{5}} g_n$$

Where $f_n = (9 + 4\sqrt{5})^{n+1} + (9 - 4\sqrt{5})^{n+1}$ and $g_n = (9 + 4\sqrt{5})^{n+1} - (9 - 4\sqrt{5})^{n+1}$, $n = 0,1,2,...$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero Distinct integer solutions are obtained as

$$x_{n+1} = 2f_n + \sqrt{5} g_n \tag{3}$$

$$y_{n+1} = \frac{1}{2\sqrt{5}} \left[\sqrt{5} f_n + 2g_n\right] \tag{4}$$

The recurrence relation satisfied by the solutions of equation (1) are given by

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0\ ;\ x_1 = 76,\ x_2 = 1364$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0\ ;\ y_1 = 17,\ y_2 = 305$$

Properties

1. $10y_{2n+2} - 2x_{2n+2} - 6$ is a square integer.
2. $6[10y_{2n+2} - 2x_{2n+2} - 6]$ is a nasty number.
3. $10y_{3n+3} - 2x_{3n+3} + 90y_{n+1} - 18x_{n+1}$ is a cubic integer.
4. $10y_{4n+4} - 2x_{4n+4} + 4t_{4,f_n} - 4$ is a bi-quadratic integer.

Remarkable Observations

By considering the linear combination among the solutions of (1), one may

Obtain different solutions of hyperbolas. Seven of them are presented in table 1 below:

Table 1: Hyperbolas

S. No.	x, y	Hyperbolas
1	$10y_{n+1} - 2x_{n+1}, \quad 5x_{n+1} - 20y_{n+1}$	$5x^2 - y^2 = 20$
2	$17x_{n+1} - x_{n+2}, \quad 5x_{n+1} - 20y_{n+1}$	$5x^2 - 16y^2 = 320$
3	$305x_{n+1} - x_{n+3}, \quad 5x_{n+1} - 20y_{n+1}$	$5x^2 - 5184y^2 = 103680$
4	$10y_{n+1} - 2x_{n+1}, \quad 19x_{n+1} - x_{n+2}$	$20x^2 - y^2 = 80$
5	$10y_{n+1} - 2x_{n+1}, \quad 341x_{n+1} - x_{n+3}$	$6480x^2 - y^2 = 25920$
6	$10y_{n+1} - 2x_{n+1}, \quad 170y_{n+1} - 10y_{n+2}$	$80x^2 - y^2 = 320$
7	$10y_{n+1} - 2x_{n+1}, \quad 3050y_{n+1} - 10y_{n+3}$	$25920x^2 - y^2 = 103680$

By considering the linear combination among the solutions of (1), one may obtain

Solutions of different parabolas. Five of them are given in table 2 below:

Table 2: Parabolas

S. No.	x, y	Parabolas
1	$10y_{2n+2} - 2x_{2n+2} + 2, \quad 5x_{n+1} - 20y_{n+1}$	$6y^2 = 5x - 120$
2	$10y_{2n+2} - 2x_{2n+2} + 2, \quad 19x_{n+1} - x_{n+2}$	$6y^2 = 20x - 480$
3	$10y_{2n+2} - 2x_{2n+2} + 2, \quad 341x_{n+1} - x_{n+3}$	$y^2 = 1080x - 25920$
4	$10y_{2n+2} - 2x_{2n+2} + 2, \quad 170y_{n+1} - 10y_{n+2}$	$6y^2 = 80x - 1920$
5	$10y_{2n+2} - 2x_{2n+2} + 2, \quad 3050y_{n+1} - 10y_{n+3}$	$y^2 = 4320x - 103680$

Choice 2:

$$t = 2k, \quad k > 0$$

The Pell equation is

$$x^2 = 20y^2 - 4^{2k}, \quad k > 0. \qquad (5)$$

Let (x_0, y_0) be the initial solutions of equation (5), given by

$$x_0 = 4^{k-1} \cdot 2, \quad y_0 = 4^{k-1} \cdot 1$$

To find the other solutions of equation (5), consider the Pellian equation

$$x^2 = 20y^2 + 1$$

Whose initial solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2} f_n$$

$$\tilde{y}_n = \frac{1}{4\sqrt{5}} g_n$$

Where $f_n = \left(9 + 4\sqrt{5}\right)^{n+1} + \left(9 - 4\sqrt{5}\right)^{n+1}$ and $g_n = \left(9 + 4\sqrt{5}\right)^{n+1} - \left(9 - 4\sqrt{5}\right)^{n+1}$, $n = 0,1,2,\ldots$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained as

$$x_{n+1} = 4^{k-1}\left[f_n + \sqrt{5}g_n\right] \tag{6}$$

$$y_{n+1} = \frac{4^{k-1}}{2\sqrt{5}}\left[\sqrt{5}f_n + g_n\right] \tag{7}$$

The recurrence relation satisfied by the solutions of equation (5) are given by

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0 \ ; \ x_1 = 4^{k-1}\cdot 58, \ x_2 = 4^{k-1}\cdot 1042$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0 \ ; \ y_1 = 4^{k-1}\cdot 13, \ y_2 = 4^{k-1}\cdot 233$$

Choice 3:

$t = 2k + 1, \quad k > 0$

The Pell equation is

$$x^2 = 20y^2 - 4^{2k+1}, \quad k > 0. \tag{8}$$

Let (x_0, y_0) be the initial solutions of equation (8), given by

$$x_0 = 4^{k-1}\cdot 4, \quad y_0 = 4^{k-1}\cdot 2$$

To find the other solutions of equation (8), consider the Pellian equation

$$x^2 = 20y^2 + 1$$

Whose initial solution $(\tilde{x}_n, \tilde{y}_n)$ is given by?

$$\tilde{x}_n = \frac{1}{2}f_n$$

$$\tilde{y}_n = \frac{1}{4\sqrt{5}}g_n$$

Where $f_n = \left(9 + 4\sqrt{5}\right)^{n+1} + \left(9 - 4\sqrt{5}\right)^{n+1}$ and $g_n = \left(9 + 4\sqrt{5}\right)^{n+1} - \left(9 - 4\sqrt{5}\right)^{n+1}$, $n = 0,1,2,\ldots$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions are obtained as

$$x_{n+1} = 2\cdot 4^{k-1}\left[f_n + \sqrt{5}g_n\right] \tag{9}$$

$$y_{n+1} = \frac{4^{k-1}}{\sqrt{5}}\left[\sqrt{5}f_n + g_n\right] \tag{10}$$

The recurrence relation satisfied by the solutions of equation (8) are given by

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0 \ ; \ x_1 = 4^{k-1}\cdot 116, \ x_2 = 4^{k-1}\cdot 2084$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0 \ ; \ y_1 = 4^{k-1}\cdot 26, \ y_2 = 4^{k-1}\cdot 466$$

4. Conclusion

One may search for other patterns of solutions to the similar equation considered above.

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