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A study on fractional calculus and their problems

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Abstract
This report is aimed at the engineering and/or scientific professional who wishes to learn about Fractional Calculus and its possible applications in his/her field(s) of study. The intent is to first expose the reader to the concepts, applicable definitions, and execution of fractional calculus (including a discussion of notation, operators, and fractional order differential equations), and second to show how these may be used to solve several modern problems. Also included within is a list of applicable references that may provide the reader with a library of information for the further study and use of fractional calculus.

Keywords: fractional calculus, applications

Introduction

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent.

According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when considering exponents of non-integer value. While almost anyone can verify how might one describe the physical meaning or moreover the transcendental exponent.

One cannot conceive what it might be like to multiply a number or quantity by itself or π times, and yet these expressions have a definite value for any value x, verifiable by infinite series expansion, or more practically, by calculator.

Now, in the same way consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally.

Given the satisfaction of a very few restrictions (e.g. function continuity), completing n integrations can become as methodical as multiplication. But the curious mind cannot be restrained from asking the question what if n were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted (pun intended), but as this report will show, fractional calculus flows quite naturally from our traditional definitions.

And just as fractional exponents such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order 1/2 and beyond can find practical use in many modern problems.

Review of Related Literature

Most authors on this topic will cite a particular date as the birthday of so called 'Fractional Calculus'. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function.

L'Hopital's posed the question to Leibniz, what would the result be if n = 1/2. Leibniz's response:" An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born. Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics.

Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences. Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus (not yet the world as a whole) are the Riemann-Liouville and Grunwald-Letnikov definition.

While the shear number of actual definitions are no doubt as numerous as the men and women that study this field, they are for the most part variations on the themes of these two and so are addressed in detail in this document. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century.

However, it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The mathematics has in some cases had to change to meet the requirements of physical reality. Caputo reformulated the more ’classic’ definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations.

As recently as 2006, Kolowankar reformulated again, the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions. Leibniz’s response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found.

However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts.

Research Study

Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts. These are The Gamma Function, The Beta Function, The Laplace Transform, and the Mittag-Leffler Function and are addressed in the following four subsections.

The Fractional Integral It was touched upon in the introduction that the formulation of the concept for fractional integrals and derivatives was a natural outgrowth of integer order integrals and derivatives in much the same way that the fractional exponent follows from the more traditional integer order exponent. For the latter, it is the notation that makes the jump seem obvious.

While one cannot imagine the multiplication of a quantity a fractional number of times, there seems no practical restriction to placing a non-integer into the exponential position. Similarly, the common formulation for the fractional integral can be derived directly from a traditional expression of the repeated integration of a function.

This approach is commonly referred to as the Riemann-Liouville approach demonstrates the formula usually attributed to Cauchy for evaluating the n th integration of the function f(t).

$$\int \cdots \int_0^t f(\tau)d\tau = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau$$

For the abbreviated representation of this formula, we introduce the operator J n such as

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau$$

Applications

Due to the summation form of the Grunwald-Letnikov definition of the fractional derivative and integral, this formula lends itself to adaptation for use in a computer numerical solver. Given a function, defined in the required function this script will compute the integer or non-integer order integral or derivative of this function over a user defined domain.

One question that arises in the programming of such a function is, because the Grunwald Letnikov definition specifies what is essentially an infinite sum, what number of these terms must be computed and summed for an accurate result to be achieved.

Because of the speed of modern computing equipment and the relative simplicity of the step by step calculation (except perhaps the calculation of the factorial), one might suppose that anywhere from 10000 to 1000000 steps would provide excellent accuracy without a significant penalty in computation time.

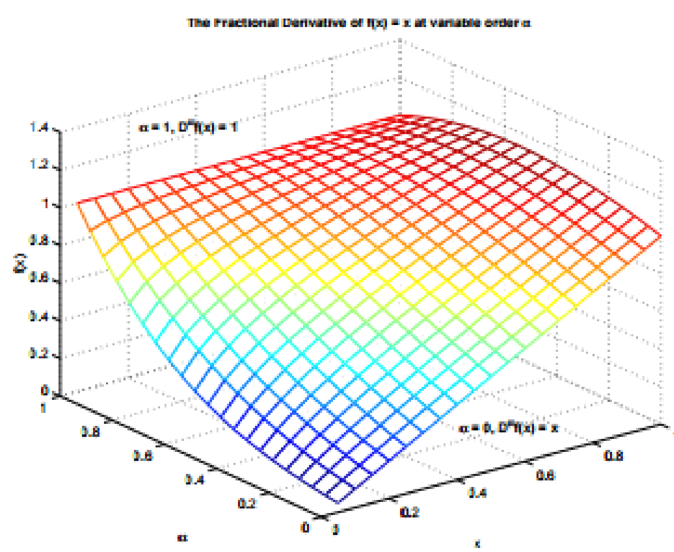
In theory, this would be a correct assumption, as even the calculation of Gamma is done approximately in most mathematical software packages including MATLAB. However the limiting factor to the accuracy of this numerical solver is not the speed of the computer. Rather it is the capacity of the computer to accurately store and move the numbers that are being supplied it.

Consider the value of Γ(z) for large values of z. At z = 171, the approximate value provided by the MATLAB gamma function is 7.26e306. At z = 172, MATLAB begins to refer to the value Γ(z) as ’Inf’. Hence, for GL differintegral.m, there is no practical use of calculating terms for the GL definition beyond the 171st.

One potential way around this software imposed upper limit is to use gamma function approximations to simplify the multiplication and division of gamma terms necessary for the GL definition. One will find, in fact, that doing so will significantly enhance the ability of MATLAB and similar software to compute higher order terms.

But what benefit does this approximation afford the user. When all terms up to the 171st are computed, the error between the analytical derivative of f(x) = x, Dα f(x) = 2px π and the numerical result of GL differintegral is on average less than 1e − 3%. Of course the calculation of more terms will lessen this error the limited summing capabilities of the MATLAB Script presented here already maintains a high level of accuracy.

The special benefits of having software that can numerically calculate the fractional derivative and integral is it provides the ability to actually examine the form of fractional order 20 calculus.



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